Tautological classes on moduli spaces of hyperkähler manifolds after Bergeron and Li

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1 Tautological conjectures

Let $\mathscr{F}_h$ be a connected component of the moduli space of pairs of h-quasi-polarised hyperkähler manifolds[1] of dimension $2n$ and with second Betti number $b_2 = b + 3$. Let $\mathscr{F}_{\Sigma,h}$ be (possibly disconnected) moduli space of h-quasi-ample Σ-polarised hyperkähler manifolds; see Olivier's notes.

As in Daniel's talk, we pretend that all the moduli space are smooth and fine. This is indeed true up to a cover; see [2, §3.4]. Hence, let $\pi_h \colon \mathscr{U}_h \to \mathscr{F}_h$ and $\pi_\Sigma \colon \mathscr{U}_{\Sigma,h} \to \mathscr{F}_{\Sigma,h}$ be the universal families, and let

$$\mathscr{B}_\Sigma = \{\mathscr{L}_1, \dots, \mathscr{L}_r\} \subset \mathrm{Pic}(\mathscr{U}_{\Sigma,h})$$

be a collection of line bundles whose image in $\mathrm{Pic}(\mathscr{U}_{\Sigma,h}/\mathscr{F}_{\Sigma,h})$ forms a basis.

We define the following subalgebras in $\mathrm{CH}^*(\mathscr{F}_h)$:

- $\mathrm{NL}^*(\mathscr{F}_h)$ is the subalgebra generated by the cycle of the connected components of the image of the maps

$$i_\Sigma \colon \mathscr{F}_{\Sigma,h} \to \mathscr{F}_h;$$

- the *tautological ring* $\mathrm{R}^*(\mathscr{F}_h)$ is the subalgebra generated by κ-cycles

$$(i_\Sigma \circ \pi_\Sigma)_*\Big(\prod_{i=1}^{r} c_1(\mathscr{L}_i)^{a_i} \prod_{j=1}^{2n} c_j(T_{\pi_\Sigma})^{b_j}\Big);$$

- the *special tautological ring* $\mathrm{DR}^*(\mathscr{F}_h)$ is the subalgebra generated by special κ-cycles

$$(i_\Sigma \circ \pi_\Sigma)_*\Big(\prod_{i=1}^{r} c_1(\mathscr{L}_i)^{a_i}\Big);$$

[1] Often we may replace quasi-polarised hyperkähler manifolds with polarised hyperkähler manifolds, since the restriction map $H^*(\mathscr{F}_{\Sigma,h}) \to H^*(\mathscr{F}_{\Sigma,h\text{-pol}})$ is surjective onto the non-zero lowest weight part, containing the image of the cycle map.

We have natural inclusions

$$\mathrm{NL}^*(\mathscr{F}_h) \subseteq \mathrm{DR}^*(\mathscr{F}_h) \subseteq \mathrm{R}^*(\mathscr{F}_h).$$

Conjecture 1.1 (Generalised Tautological Conjecture)**.**

$$\mathrm{NL}^*(\mathscr{F}_h) = \mathrm{R}^*(\mathscr{F}_h).$$

We add the subscript hom to denote the image of the corresponding ring in $H^*(\mathscr{F}_h, \mathbb{Q})$ via the cycle class map.

Conjecture 1.2 (Cohomological Tautological Conjecture)**.**

$$\mathrm{NL}^*_{\mathrm{hom}}(\mathscr{F}_h) = \mathrm{R}^*_{\mathrm{hom}}(\mathscr{F}_h). \tag{1}$$

Theorem 1.3. [2, Thm. 4.3.1] *If $b_2 \geq 6$, then*

$$\mathrm{NL}^*_{\mathrm{hom}}(\mathscr{F}_h) = \mathrm{DR}^*_{\mathrm{hom}}(\mathscr{F}_h). \tag{2}$$

If $b_2 > 8n$ (e.g. $K3^{[n]}$ for $n \leq 2$), then

$$\mathrm{NL}^*_{\mathrm{hom}}(\mathscr{F}_h) = \mathrm{R}^*_{\mathrm{hom}}(\mathscr{F}_h).$$

If $b_2 \geq 6$ and the very general fiber of π_h admits a multiplicative Chow–Künneth decomposition and the Beauville–Voisin conjecture holds for it, then up to shrinking to an open subset of $\mathscr{F}_h$, we still have equality (1)*.*

Consider the local systems $\mathbb{H}^j := R^j\pi_{h,*}\mathbb{Q}_{\mathscr{U}_h}$. The degeneration of the Leray spectral sequence for π_h gives the morphism of mixed Hodge structures

$$H^d(\mathscr{U}_h, \mathbb{Q}) \simeq \bigoplus_{i+j=d} H^i(\mathscr{F}_h, \mathbb{H}^j). \tag{3}$$

Using this decomposition the push-forward $\pi_{h,*}$ in cohomology just becomes the projection onto the factor $H^i(\mathscr{F}_h, \mathbb{H}^{4n})$.

One can consider Noether–Lefschetz cycles with coefficients in a local system $\mathbb{E}$. Indeed, given an irreducible Noether–Lefschetz cycle $Z \subset \mathscr{F}_{\Sigma,h}$ of codimension k there is a Gysin map

$$H^0(Z, \mathbb{E}|_Z) \to H^{2k}(\mathscr{F}_h, \mathbb{E}),$$

which is the analogue of the Gysin map for cohomology with coefficients in a local system; see [9, Thm. B.36]. The subspace $\mathrm{NL}^*_{\mathrm{hom}}(\mathscr{F}_{\Sigma,h}, \mathbb{E}) \subseteq H^*(\mathscr{F}_{\Sigma,h}, \mathbb{E})$ is then defined to be the subalgebra generated by the images of all such maps.

The proof of Theorem 1.3 then relies on the following properties of the cohomology of $\mathscr{F}_{\Sigma,h}$ with coefficients in $\mathbb{H}^j$ (which are of independent interest!).

Theorem 1.4 (Cohomology of $\mathscr{F}_{\Sigma,h}$)**.** *The following facts hold.*

1. *(Purity) The mixed Hodge structure of $H^{<b-1}(\mathscr{F}_{\Sigma,h}, \mathbb{E})$ is pure.*
2. *(Vanishing of low odd cohomology) If $2d+1 < b/2$, $H^{2d+1}(\mathscr{F}_{\Sigma,h}, \mathbb{E}) = 0$.*
3. *(Algebraicity of the low and high cohomology)*
$$H^{d,d}(\mathscr{F}_{\Sigma,h}, \mathbb{E}) = \mathrm{NL}^d_{\mathrm{hom}}(\mathscr{F}_{\Sigma,h}, \mathbb{E}) \text{ for } d < \frac{b+1}{3},$$
$$(W_{2d}H^{2d}(\mathscr{F}_{\Sigma,h}, \mathbb{E}))^{d,d} = \mathrm{NL}^d_{\mathrm{hom}}(\mathscr{F}_{\Sigma,h}, \mathbb{E}) \text{ for } d > \frac{2b-1}{3}.$$
4. *(Hodge–Tate low cohomology)*
$$H^{2d}(\mathscr{F}_{\Sigma,h}, \mathbb{E}) = H^{d,d}(\mathscr{F}_{\Sigma,h}, \mathbb{E}) \text{ for } d < \frac{b}{4}.$$
5. *(Vanishing of top algebraic cohomology) If $b > 3$, $H^{\geq 2b-2}_{\mathrm{alg}}(\mathscr{F}_{\Sigma,h}, \mathbb{E}) = 0$.*

Question 1.5. *Is the mixed Hodge structure on $H^*(\mathscr{F}_h, \mathbb{Q})$ pure?*

Corollary 1.6 (Cohomological Franchetta Conjecture)**.** *A cycle $\alpha \in H^{<b/2}_{\mathrm{alg}}(\mathscr{U}_h, \mathbb{Q})$ whose restriction to the very general fibre is homologous to zero is supported over proper Noether–Lefschetz loci in $\mathscr{F}_h$.*

Proof. By Theorem 1.4.(3)-(4) we have
$$\alpha \in H^{<b/2}_{\mathrm{alg}}(\mathscr{U}_h) \subseteq \bigoplus_{i+j<b/4, i\neq 0} H^{2i}(\mathscr{F}_h, \mathbb{H}^{2j}) = \bigoplus_{i+j<b/4, i\neq 0} \mathrm{NL}^i_{\mathrm{hom}}(\mathscr{F}_h, \mathbb{H}^{2j}).$$
□

Proof of Theorem 1.3 assuming Theorem 1.4. The proof can be divided in several steps.

Step 1. *In order to prove* (1)*, it is enough to show that*
$$\prod_{i=1}^{r} c_1(\mathscr{L}_i)^{a_i} \prod_{j=1}^{2n} c_j(T_{\pi_h})^{b_j} \in \mathrm{NL}_{\mathrm{hom}}(\mathbb{H}) := \bigoplus_{i=0}^{2n} \mathrm{NL}^{2i}_{\mathrm{hom}}(\mathscr{F}_h, \mathbb{H}^{4n-2i}), \qquad (4)$$
since its pushforward lies in $\mathrm{NL}^*_{\mathrm{hom}}(\mathscr{F}_h)$.

Step 2. *It is enough to show that* (4) *holds for $c_1(\mathscr{L}_i)$ and for $c_j(T_{\pi_h})$ if $j < b/8$* (or equivalently $b_2 > 8n$). Indeed, given $\alpha_l \in H^{<b/4}_{\mathrm{alg}}(\mathscr{U}_h)$, with $l = 1, \ldots, L$, we have that $\alpha_1 \wedge \ldots \wedge \alpha_L \in H^*(\mathscr{U}_h)$ lies in $\mathrm{NL}_{\mathrm{hom}}(\mathbb{H})$.

To see this, note that the cup product on the LHS of (3) induces a cup product on the Leray spectral sequence, and so a cup product on the RHS of (3); see

[10, Lemme 16.13]. The isomorphism (3) however may not preserve the ring structure; see for instance [11, Prop. 0.4 or Prop. 0.6]. Now Funke–Kudla–Millson describe a special subring $\Phi_{\mathbf{H}^*}$ of the RHS of (4) closed by definition under RHS cup products and such that

$$\mathrm{NL}^{<(b+1)/3}_{\mathrm{hom}}(\mathscr{F}_h, \mathbb{H}^j) \subseteq \Phi_{\mathbf{H}^*} \subseteq \mathrm{NL}^*_{\mathrm{hom}}(\mathscr{F}_h, \mathbb{H}^j);$$

see Proposition 6.1.

Claim 1.7 (Theorem 8.2.1)**.** *Let $\alpha_l \in H^{<b/4}_{\mathrm{alg}}(\mathscr{U}_h)$, with $l = 1, \ldots, L$. Then*

$$\alpha_1 \wedge_{\mathrm{LHS}} \cdots \wedge_{\mathrm{LHS}} \alpha_L - \alpha_1 \wedge_{\mathrm{RHS}} \cdots \wedge_{\mathrm{RHS}} \alpha_L$$

is supported over Noether–Lefschetz loci of $\mathscr{F}_h$.

Proof. The last four lines of the proof of Theorem 8.2.1 are not clear to us. We have emailed Bergeron in this regard. □

Step 3. *$c_1(\mathscr{L}_i)$ and $c_j(T_{\pi_h})$ lie in $\mathrm{NL}_{\mathrm{hom}}(\mathbb{H})$ if $j < b/4$.* This follows from Theorem 1.4.(3).

Step 4. *Eq.* (2) *actually holds for the sharper bound $b > 2$.* Indeed, if $b > 2$, then $H^1(\mathscr{F}_h, \mathbb{H}^1) = 0$ by Theorem 1.4.(2), and so we have

$$c_1(\mathcal{L}_i) \in H^0(\mathscr{F}_h, \mathbb{H}^2) \oplus H^{1,1}(\mathscr{F}_h, \mathbb{Q}) = H^0(\mathscr{F}_h, \mathbb{H}^2) \oplus \mathrm{NL}_{\mathrm{hom}}(\mathscr{F}_h, \mathbb{Q})$$

due to Theorem 1.4.(3). Up to subtracting the pull-back of a Noether–Lefschetz divisor, we can suppose $c_1(\mathcal{L}_i) \in H^0(\mathscr{F}_h, \mathbb{H}^2)$. Therefore, up to a Noether–Lefschetz cycle (cf. Claim 1.7), we have

$$\prod_{i=1}^{r} c_1(\mathscr{L}_i)^{a_i} \in H^0(\mathscr{F}_h, \mathbb{H}^{\sum_{i=1}^r 2a_i}) = \mathrm{NL}^0_{\mathrm{hom}}(\mathscr{F}_h, \mathbb{H}^{\sum_{i=1}^r 2a_i})$$

by Theorem 1.4.(3). This yields Eq. (2).

Step 5. *We show that Eq.* (1) *holds modulo the existence of a multiplicative Chow–Künneth decomposition and the Beauville–Voisin conjecture.* Let X be the very general fiber of $\pi_h \colon \mathscr{U}_h \to \mathscr{F}_h$ of dimension $2n$. The existence of a multiplicative Chow–Künneth decomposition implies that the diagonal $\Delta_X \in \mathrm{CH}^{2n}(X \times X)$ admits a decomposition

$$\Delta_X = \Pi_0 + \cdots + \Pi_{4n} \in \mathrm{CH}^{2n}(X \times X),$$

where Π_i are orthogonal projectors, i.e. $\Pi_i \circ \Pi_i = \Pi_i$ and $\Pi_j \circ \Pi_i = 0$ for $i \neq j$. This yields a bigrading of the Chow ring $\mathrm{CH}^i(X)_s \coloneqq \Pi_{2i-s}\,\mathrm{CH}^i(X)$. Here, the projectors act via the calculus of correspondences. The intersection product respects the bigrading.

Moreover, the decomposition should conjecturally give a splitting of the conjectural Bloch–Beilinson filtration of the Chow groups (if, strictly speaking, one furthermore assumes that $\mathrm{CH}^i(X)_s = 0$ for $s < 0$ and the subalgebra $\mathrm{CH}^i(X)_s = 0$ for $s > 0$ is exactly the kernel of the cycle class map).

The Beauville–Voisin conjecture asserts that the first Chern class of line bundles $c_1(\mathscr{L})$ and the Chern classes of the tangent bundle $c_j(T_X)$ are contained in the subring $\mathrm{CH}^*(X)_0$. This means that the projector Π_{2k} acts as the identity on $\prod_{i=1}^r c_1(\mathscr{L}_i)^{a_i} \prod_{j=1}^{2n} c_j(T_{\pi_\Sigma})^{b_j}$, with $\sum_i a_i + \sum_j jb_j = k$, and all other projectors Π_i act trivially.

By spreading out, there exists a (non-surjective) étale chart of $\mathscr{F}'_h \to \mathscr{F}_h$ with open image such that $\pi'_h \colon \mathscr{U}'_h \coloneqq \mathscr{U}_h \times_{\mathscr{F}'_h} \mathscr{F}_h \to \mathscr{F}'_h$ has a multiplicative Chow–Künneth decomposition of the diagonal $\Delta_{\mathscr{U}'_h} = \sum \Pi'_i$. The relative projectors Π'_j viewed in cohomology induce a splitting as in (3) [11, Lem. 2.1]. The Beauville–Voisin conjecture now implies that up to further shrinking the chart $\mathscr{F}'_h$ one has that the Chern classes $c_j(T_{\pi'_h})$ lie in $H^0(\mathscr{F}'_h, \mathbb{H}^{2j})$. Theorem 1.4.(3) then gives that all Chern classes $c_j(T_{\pi'_h})$ are Noether–Lefschetz. Since $c_j(T_{\pi'_h})$ is the preimage of $c_j(T_{\pi_h})$, the same holds for the Chern classes $c_j(T_{\pi_h})$. □

Remark 1.8 (Limit of the approach of Bergeron–Li)**.** It seems unclear how the methods of Bergeron–Li can show that the (i)-component of $c_j(T_{\pi_h})$ in the RHS of (3) is in $\mathrm{NL}_{\mathrm{hom}}(\mathbb{H})$ when $\frac{b}{2} \leq i \leq \min\{2k, 2b - \frac{b}{2}\}$.

Open problem. Show that $\pi_{h,*}c_j(T_{\pi_h})$ lies in $\mathrm{NL}^*_{\mathrm{hom}}(\mathscr{F}_h)$ for $j \geq b/4$.

2 Period domain

The main idea of Bergeron–Li is to translate CTC in a statement about special cycles of Shimura varieties, and to exploit the modularity of generating functions whose coefficients are intersection numbers of special cycles.

We briefly recall the notion of period domain; see also Olivier's notes.

- Λ is a lattice with signature $(2, b)$;
- $SO(\Lambda_\mathbb{R}) \simeq SO(2, b)$ is the special orthogonal group of a quadratic form with signature $(2, b)$;
- $SO^+(2, b)$ is the connected component of the identity of $SO(2, b)$.
- The *period domain* is
$$\begin{aligned}\hat{D} &\coloneqq \{[\omega] \in \mathbb{P}(\Lambda_\mathbb{C}) \mid \omega^2 = 0, \omega \cdot \bar{\omega} > 0\}\\ &\simeq \{\text{positive oriented 2-planes in } \Lambda_\mathbb{R}\}\\ &\simeq SO(2, b)/SO(2) \times SO(b).\end{aligned}$$
- A connected component of the period domain
$$D \coloneqq SO^+(2, b)/SO(2) \times SO(b)$$

is a contractible Hermitian symmetric domain of complex dimension b.

- Any torsion-free arithmetic subgroup $\Gamma \subset SO^+(\Lambda)$ acts properly discontinuously on D with quasi-projective quotient $\Gamma\backslash D$.
- $\Sigma_{\text{prim}} \subseteq \Lambda$ is a negative-definite sub-lattice of rank r;
- D_Σ is the subset of 2-planes orthogonal to Σ_{prim} in D;
- $\Gamma_\Sigma \coloneqq \{g \in \Gamma \mid g(\sigma) = \sigma \text{ for all } \sigma \in \Sigma_{\text{prim}}\}$.

A *special cycle* is a connected cycle $c(\Sigma, \Gamma) \in \mathrm{CH}^r(\Gamma\backslash D)$, given by the image of the morphism

$$\Gamma_\Sigma\backslash D_\Sigma \to \Gamma\backslash D.$$

Definition 2.1. $\mathrm{SC}^r(\Gamma\backslash D)$ *is the subgroup spanned by special cycles* $c(\Sigma, \Gamma)$ *of codimension* r*, and set*

$$\mathrm{SC}^*(\Gamma\backslash D) \subset \mathrm{CH}^*(\Gamma\backslash D)$$

to be the subalgebra generated by special cycles of any codimension.

The following facts hold.

1. The subgroup spanned by special cycles coincides with the subalgebra generated by them

$$\mathrm{SC}^*(\Gamma\backslash D) = \bigoplus_r \mathrm{SC}^r(\Gamma\backslash D).$$

Proof. Take two special cycles $\alpha \in \mathrm{SC}^l(\Gamma\backslash D)$ and $\beta \in \mathrm{SC}^k(\Gamma\backslash D)$.

If $l = 1$, there are two cases. Firstly, the cycles α and β intersect properly, in which case the result is immediate. So let us assume that β is contained in α and that α is irreducible. Generalising the discussion from Daniel's notes (page 10) to the hyperkähler setting, consider the short exact sequence[2]

$$0 \to \mathcal{T}_{\mathscr{F}_h} \to R^1\pi_{h,*}\mathcal{T}_\pi \to R^2\pi_{h,*}\mathcal{O} \to 0 \tag{5}$$

and note that it also holds for $\pi_\Sigma \colon \mathscr{U}_{\Sigma,h} \to \mathscr{F}_{\Sigma,h}$. It follows from a calculation analogous to the one Daniel performed (see also [2, Ex. 4.4]) that $\mathrm{c}_1(\mathcal{T}_{\mathscr{F}_{\Sigma,h}}) = -(b + 1 - \mathrm{rank}\Sigma)\lambda$, where $\lambda \coloneqq c_1(R^0\pi_{\Sigma,*}\Omega_{\mathscr{U}_{\Sigma,h}/\mathscr{F}_{\Sigma,h}})$ is the Hodge bundle. If α corresponds to Σ, we conclude that the first Chern class of the normal bundle of Σ is contained in $\langle\lambda\rangle$[3]. In particular, $\alpha \cdot \beta \in \langle\lambda \cdot \beta\rangle$.

[2]The first map is the relative Kodaira-Spencer map, the second map is the contraction by the class h. See for instance [8, p.25].

[3]Alternatively, note that a special divisor is the image in $\Gamma\backslash D$ of the intersection of D with a hyperplane $H \subset \mathbb{P}(\Lambda_{\mathbb{C}})$. Up to multiples, $\mathcal{O}_{H\cap D}(1)$ descends to the normal bundle of a Noether–Lefschetz divisor. On the other hand, the tautological bundle $\mathcal{O}_D(-1)$ descends to the Hodge bundle.

The case $l > 1$ follows from a splitting principle-type argument for connected cycles. $\square$

2. The period map[4] $\mathscr{P}_h \colon \mathscr{F}_h \to \Gamma\backslash D$ sends Noether–Lefschetz cycles to special cycles
$$\mathscr{P}_h^*(\mathrm{NL}^*(\mathscr{F}_h)) = \mathrm{SC}^*(\Gamma\backslash D).$$

For the rest of the talk we are going to hint at the proof and the ideas behind Theorem 1.4. In view of the last remark, we can rephrase the theorem in terms of special cycles. For simplicity, we will work mainly with the trivial local system $\mathbb{E} = \mathbb{Q}_{\Gamma\backslash D}$.

3 Zucker's conjecture

In order to study the Hodge theory of open or singular varieties, intersection cohomology of a compactification and L^2-cohomology are often more indicate than singular cohomology. Actually it is conjectured that they should be isomorphic in many cases; see for instance [7, p. 88]. One remarkable case where the conjecture is known to hold is that of locally symmetric varieties like $\Gamma\backslash D$.

Let $\Gamma\backslash D^*$ be the Baily-Borel-Satake compactification of $\Gamma\backslash D$. Recall that it is a projective compactification whose boundary has dimension one. The following conjecture have been proved by Looijenga, Saper and Stern.

Theorem 3.1 (Zucker's conjecture). $H^*_{(2)}(\Gamma\backslash D, \mathbb{Q}) \simeq IH^*(\Gamma\backslash D^*, \mathbb{Q})$.

1. $IH^*(\Gamma\backslash D^*, \mathbb{Q})$ carries a pure Hodge structure, since it is a mixed sub-Hodge structure of the cohomology of the resolutions of singularities of $\Gamma\backslash D^*$.
2. $H^*_{(2)}(\Gamma\backslash D, \mathbb{Q})$ carries a pure Hodge structure, since it is spanned by harmonic forms as $\Gamma\backslash D$ is complete and $H^*_{(2)}(\Gamma\backslash D, \mathbb{Q})$ is finite dimensional; see also Chern's theorem below.
3. A priori the isomorphism in Zucker's conjecture is not a morphism of Hodge structures, but the natural map
$$\xi \colon H^*_{(2)}(\Gamma\backslash D, \mathbb{Q}) \simeq IH^*(\Gamma\backslash D^*, \mathbb{Q}) \to H^*(\Gamma\backslash D, \mathbb{Q})$$
is a morphism of Hodge structures.
4. $$\mathrm{SC}^*_{\mathrm{hom}}(\Gamma\backslash D) \subseteq W_* H^*(\Gamma\backslash D, \mathbb{Q}) = \xi(H^*_{(2)}(\Gamma\backslash D^*, \mathbb{Q})). \tag{6}$$
5. Further, we have
$$H^{<b-1}_{(2)}(\Gamma\backslash D, \mathbb{Q}) \simeq IH^{<b-1}(\Gamma\backslash D^*, \mathbb{Q}) \simeq H^{<b-1}(\Gamma\backslash D, \mathbb{Q}). \tag{7}$$

[4]See Olivier's notes for a definition.

If $\Gamma\backslash D^*$ was smooth, we would have had equality for $< 2b-1$ by Thom's isomorphism. The bound in the singular case is smaller and depends on the cohomology of the link of the singularities; see for instance from [6, Lemma 1]. This gives the purity in Theorem 1.4.(1).

4 Relative Lie algebra cohomology

We recall some classical facts about the cohomology of Lie groups. We fix some notation.

- G is a real Lie group with Lie algebra $\mathfrak{g} = T_e G$;
- E is a G-representation with $\mathfrak{g}$-module structure $\rho\colon \mathfrak{g} \to \mathrm{GL}(E)$;
- K is a connected closed subgroup of G;
- the local system $\mathbb{E} := (G \times E)/K$ is the suspension of E.

For $g \in G$, the left translation by g^{-1} provides a canonical identification of $T_g G$ with $\mathfrak{g}$. The complex of smooth differential forms $(A^*(G;E), d)$ with values in E can be identified with $(\mathrm{Hom}(\Lambda^*\mathfrak{g}, \mathscr{C}^\infty(G) \otimes E), \delta)$. Note that the differential

$$\begin{aligned}\delta\omega(X_1,\dots,X_k) &= \sum(-1)^i \rho(X_i)(\omega(X_1,\dots,\widehat{X}_i,\dots,X_k)) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k)\end{aligned}$$

depends only on $\mathfrak{g}$ and the $\mathfrak{g}$-module structure of $V := \mathscr{C}^\infty(G) \otimes E$. The cohomology of this complex is called the cohomology of $\mathfrak{g}$ with values[5] in V, and it is denoted $H^*(\mathfrak{g}; V)$. Analogously, given $\pi : G \to G/K$, we have

$$\begin{aligned}(A^*(G/K;E),d) &\simeq \{\omega \in A^*(G;E) \mid \quad i_X\omega = 0 \text{ if } d\pi(X) = 0; \quad r_k^*\omega = \omega \ \ \forall k \in K\} \\ &= \{\omega \in \mathrm{Hom}(\Lambda^*(\mathfrak{g}/\mathfrak{k}), \mathscr{C}^\infty(G) \otimes E) \mid Ad(k)^*\omega = \omega \ \ \forall k \in K\} \\ &= \{\omega \in \mathrm{Hom}(\Lambda^*(\mathfrak{g}/\mathfrak{k}), \mathscr{C}^\infty(G) \otimes E) \mid ad(k)^*\omega = 0 \ \ \forall k \in \mathfrak{K}\} \\ &=: \mathrm{Hom}^{\mathfrak{k}}(\Lambda^*(\mathfrak{g}/\mathfrak{k}), \mathscr{C}^\infty(G) \otimes E).\end{aligned}$$

The cohomology of this complex is called the cohomology of $\mathfrak{g}$ relative to K with values in V, and it is denoted $H^*(\mathfrak{g},\mathfrak{k};V)$.

Remark 4.1. Assume G to be compact and connected, E to be finite dimensional and acted upon trivially by G. Then an averaging argument shows that $(A^*(G/K;E),d)$ is quasi-isomorphic to the complex of left G-invariant $(A^*(G/K;E)^G, d)$, and so

$$H^*_{\mathrm{dR}}(G/K;\mathbb{E}) \simeq H^*((A^*(G/K;E)^G,d)) \simeq H^*(\mathrm{Hom}^{\mathfrak{k}}(\Lambda^*(\mathfrak{g}/\mathfrak{k}),E),\delta)) = H^*(\mathfrak{g},\mathfrak{k};E).$$

However, if G is not compact, then convergence issues in the average process arise.

[5] In general, Lie algebra cohomology takes values in $(\mathfrak{g}, K)$- or $(\mathfrak{g}, \mathfrak{k})$-modules; see [5, Ch. 0].

Suppose now that Γ is a torsion-free arithmetic subgroup of G, and $\mathbb{E}$ descends to a homonymous local system on $\Gamma\backslash G/K$. Then we obtain

$$H^*(\Gamma\backslash G/K;\mathbb{E}) \simeq H^*(\mathfrak{g},\mathfrak{k};\mathscr{C}^\infty(\Gamma\backslash G)\otimes E)$$
$$H^*_{(2)}(\Gamma\backslash G/K;\mathbb{E}) \simeq H^*(\mathfrak{g},\mathfrak{k};L^2(\Gamma\backslash G)\otimes E).$$

The advantage to work with L^2-cohomology is that, according to Matsushima's formula, $L^2(\Gamma\backslash G)$ decomposes into a Hilbert direct sum of irreducible representations of G with finite multiplicity, and it induces the splitting

$$H^*_{(2)}(\Gamma\backslash G/K;\mathbb{E}) \simeq \bigoplus_\pi m(\pi)H^*(\mathfrak{g},\mathfrak{k};\pi\otimes E), \tag{8}$$

where π runs over over all the unitary representations of G with non-zero cohomology occurring in the discrete spectrum of $L^2(\Gamma\backslash G)$ with multiplicity $m(\pi)$, already classified by Vogan and Zuckerman. Hence, the results in Section 1 can be considered a manifestation of the constraints and the symmetries enjoyed by these representations. In the next section we show how these representations inform us about the Hodge theory of $\Gamma\backslash D$.

5 Refined Hodge decomposition

Theorem 5.1 (Chern's generalization of Kähler geometry)**.** *Let M be a compact Riemannian manifold of real dimension m, and let $\mathrm{Hol}^\circ(M)$ be the identity component of the holonomy group. The space of harmonic forms on M*

$$\mathcal{H}^k_{(2)}(M) \simeq \bigoplus_W \mathcal{H}^k_W(M) \tag{9}$$

splits in the direct sum of harmonic forms of type W, where W runs over the irreducible $\mathrm{Hol}^\circ(M)$-invariant subspaces of $\Lambda^*T_p(M)$ for some $p\in M$.

Note that if (M,ω) is a compact Kähler manifolds of dimension $m=2m'$, then $\mathrm{Hol}^\circ(M)\subset U(m')$. The decomposition (9) is the Lefschetz decomposition.

We turn now to the case $M=\Gamma\backslash D$. Note that $\mathrm{Hol}^\circ(M)=K$ acts on $T_p(\Gamma\backslash D)\subset \mathfrak{g}$ via the adjoint representation. By Theorem 5.1[6] there exists a decomposition of $H^*_{(2)}(M,\mathbb{C})$ whose direct summands corresponds to irreducible K-submodules of $\mathfrak{p}\coloneqq T_p(\Gamma\backslash D)\otimes\mathbb{C}$.

Hodge decomposition. K acts on $\mathfrak{p}=((\mathbb{C})^2)^\vee\otimes\mathbb{C}^b$ through the standard representation of $SO(2)$ on $\mathbb{C}^2$ and the standard representation of $SO(b)$ on $\mathbb{C}^b$. $SO(2)$ decomposes $\mathbb{C}^2$ into irreducibles

$$\mathbb{C}^2=\mathbb{C}^+\oplus\mathbb{C}^-=\mathbb{C}\langle e_1+ie_2\rangle\oplus\mathbb{C}\langle e_1-ie_2\rangle.$$

[6] In [3, §2.5] and in [4, §1.2.2] the authors suggested that Chern's result holds for non-compact locally symmetric spaces too. See also [3, Eq. (4.3)].

This induces the decompositions

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^- \qquad (T_p(\Gamma\backslash D) \otimes \mathbb{C} = T_p^{(1,0)}(\Gamma\backslash D) \oplus T_p^{(0,1)}(\Gamma\backslash D))$$
$$\Lambda^d \mathfrak{p} = \bigoplus_{r+s=d} \Lambda^r \mathfrak{p}^+ \otimes \Lambda^s \mathfrak{p}^- \qquad \text{(Hodge decomposition)}.$$

Lefschetz decomposition. The *Euler form* $\mathfrak{e}_2$ is a generator of

$$[\Lambda^2 \mathfrak{p}]^{SL(2)\times SO(b)} = [\mathrm{Sym}^2 \mathbb{C}^b]^{SO(b)} \otimes \Lambda^2(\mathbb{C}^2)^\vee = [\mathrm{End}(\mathbb{C}^b)]^{SO(b)} \otimes \Lambda^2(\mathbb{C}^2)^\vee \simeq \mathbb{C}$$
$$\mathfrak{e}_2 \coloneqq \frac{1}{2i} \sum v_\alpha \otimes (e_1 + ie_2) \wedge v_\alpha \otimes (e_1 - ie_2) \mapsto \mathrm{Id}_{\mathbb{C}^b} \otimes e_1 \wedge e_2,$$

where v_α is an orthonormal basis of $\mathbb{R}^b \subset \mathbb{C}^b$, which we can identify with a positive defined vector space in $\Lambda_{\mathbb{R}}$. The left G-invariant 2-form dual to $\mathfrak{e}_2$ descends (up to scalar) to the first Chern class of the Hodge bundle $\mathcal{O}(-1)$ (or of the canonical bundle by Eq. (5)); see [3, §2.3] and [4, §13.2]. Hence, the Lefschetz decomposition with respect to $\mathfrak{e}_2$

$$\Lambda^r \mathfrak{p}^+ \otimes \Lambda^s \mathfrak{p}^- = \bigoplus_{k=0}^{\min\{r,s\}} \tau_{r-k,s-k}.$$

is K-invariant, and in cohomology it corresponds to the Lefschetz decomposition with respect to $\mathcal{O}(1)$.

Refined Hodge decomposition. The K-representations $\tau_{r,s}$ are not irreducible in general. However, the following facts hold.

1. $\tau_{r,s}$ is irreducible for $r + s < \frac{b}{2}$;
2. (Hodge-Tate) $H^{r,s}(\mathfrak{g}, \mathfrak{k}; \pi \otimes E) = 0$ for π as in (8), if $r \neq s, r + s < \frac{b}{2}$;
3. $\tau_{r,r}$ is irreducible for $r < b$;
4. the G-representations π in (8) such that $H^{r,r}(\mathfrak{g}, \mathfrak{k}; \pi \otimes E) \neq 0$ for $r < b$ are classified and denoted $A_{r,r}(E)$; see [4, §5.4].

Together with Eq. (6) and Eq. (8), these facts imply Theorem 1.4.(2) and 1.4.(4).

6 Modularity of generating functions

The algebricity results in Theorem 1.4 relies on the modularity[7] of certain generating functions. Indeed, there exist modular forms of weight $b/2 + 1$

$$\theta_{2r}(\varphi)\colon \mathrm{Mp}_{2r} \to W_{2r} H^{2r}(\Gamma\backslash D, \mathbb{C})$$
$$g' \mapsto [\theta_{2r}(g', \varphi)].$$

[7]The modularity has been used for enumerative purposes by Maulik and Pandharipande (cf Emre's talk), or recently to bound the irrationality degree of $\mathscr{F}_h$ for K3 surfaces in [1].

For any $\eta \in H^{2b-2r}(\Gamma\backslash D, \mathbb{C})$, the Fourier expansion of the Poincaré pairing is

$$\langle \theta_{2r}(g',\varphi), \eta\rangle = \sum_{\beta \in \mathrm{Sym}_{r\times r}(\mathbb{Q})} \langle D_{\beta,\varphi} \wedge \mathcal{O}(1)^{r-\mathrm{rank}\beta}, \eta\rangle w_\beta(g')$$

for certain explicit functions $w_\beta(g')$ and special cycles $D_{\beta,\varphi}$ generalising Heegner divisors; see Olivier and Emre's notes, [3, §2.6, Proposition 8.3] and [4, Proposition 10.1] for the generalization to arbitrary $\mathbb{E}$.

The modular forms depends on a parameter φ, which is a K-invariant Schwartz form. In Emre's talk this choice correspond to a linear combination of characteristic function in the discriminant lattice $M^\vee/M$. Funke–Kudla–Millson restricted their choice of φ to a special ring Φ. Define the FKM-ring

$$\Phi_{\mathbf{H}^*} \subseteq \bigoplus_{i,j} H^i(\Gamma\backslash D, \mathbb{H}^j)$$

generated by $[\theta_{2r}(g',\varphi)] \in \bigoplus_{i,j} H^i(\Gamma\backslash D, \mathbb{H}^j)$ as g' and φ varies in Mp_{2r} and Φ.

Proposition 6.1. *The following facts hold.*

1. $\Phi_{\mathbf{H}^*}$ *is closed under the induced cup product on the RHS of* (3)*;*
2. $H^{i,i}(\mathfrak{g},\mathfrak{k}; A_{r,r}(\mathbb{H}^j)) \subseteq \Phi_{\mathbf{H}^*}$ *for* $3r < b+1$*;*
3. $\Phi_{\mathbf{H}^*} \subseteq \mathrm{SC}_{\mathrm{hom}}(\mathbb{H})$.

Sketch of the proof of Proposition 6.1.(3). It is enough to prove $\mathrm{SC}_{\mathrm{hom}}(\mathbb{H})^\perp \subset \Phi_{\mathbf{H}^*}^\perp$ with respect to Poincaré duality. Let $\eta \in \mathrm{SC}_{\mathrm{hom}}(\mathbb{H})^\perp$. Then $\langle D_{\beta,\varphi} \wedge \mathcal{O}(1)^{r-\mathrm{rank}\beta}, \eta\rangle = 0$ by hypothesis, and so $\langle \theta_{2r}(g',\varphi), \eta\rangle = 0$. Hence, $\eta \in \Phi_{\mathbf{H}^*}^\perp$.

In fact, one should prove that $\mathcal{O}(1) \in \mathrm{SC}^*_{\mathrm{hom}}(\Gamma\backslash D)$. Suppose it is not, then there exists $\eta \in H_c^{2b-2}(\Gamma\backslash D)$ such that $\langle \mathcal{O}(1), \eta\rangle \neq 0$, but η vanishes against any special cycles. We obtain that the modular form $\langle \theta_{2r}(\cdot,\varphi), \eta\rangle$ of weight $b/2+1$ is constant, equal to $\langle \mathcal{O}(1), \eta\rangle$. This contradicts the modularity! □

We obtain the following result concerning the algebraic cohomology of $\Gamma\backslash D$.

Proof of Theorem 1.4.(3) *and 1.4.*(5)*.* We have

$$H^{d,d}_{(2)}(\Gamma\backslash D, \mathbb{E}) = \bigoplus_{r<b} m(\pi) H^{d,d}(\mathfrak{g},\mathfrak{k}; A_{r,r}(E)\otimes E) \subset \mathrm{SC}_{\mathrm{hom}}(\mathbb{E}),$$

where the first equality follows from Eq. (8) and Section 5, and the last inclusion holds for $3r < b+1$ by an analogue of Proposition 6.1 for cohomology with coefficients in the local systems $\mathbb{E}$. This gives the first statement of Theorem 1.4.(3).

Hard Lefschetz for intersection cohomology implies

$$H^{d,d}_{(2)}(\Gamma\backslash D, \mathbb{E}) = IH^{d,d}(\Gamma\backslash D, \mathbb{E}) \xrightarrow[\simeq]{\cup\mathcal{O}(1)^{b-2d}} IH^{b-d,b-d}(\Gamma\backslash D, \mathbb{E}) \qquad (10)$$

$$\twoheadrightarrow (W_{2(b-d)}H^{2(b-d)}(\Gamma\backslash D, \mathbb{E}))^{d,d},$$

and it shows the second statement of Theorem 1.4.(3). In particular, since $\mathcal{O}(1)^{b-2} \cdot c(\Sigma, \Gamma) = 0$ (see Daniel's talk), the composition (10) is the zero map for $d = 1$, thus we get Theorem 1.4.(5). ☐

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