PICARD GROUP AND NL-CYCLES OF $\mathcal{F}_g$

FABRIZIO ANELLA

Abstract. These notes are mainly devoted to the following conjecture of Maulik and Pandharipande [6].

Conjecture 0.1. *The rational Picard group of the moduli space of quasi-polarized K3 surfaces is generated by NL-divisors.*

Then we discuss two problems that are naturally related to this conjecture:

Problem 0.2. *Give an explicit basis of the rational Picard group of quasi-polarized K3 surfaces.*

Problem 0.3. *Describe the rational cohomology classes of the moduli space of quasi-polarized K3 surfaces spanned by NL-cycles.*

1. Preliminary

1.1. **The moduli space.** Let $g \geq 2$ be an integer and (S, L) a primitive quasi-polarized K3 surface of genus g, *i.e.* L is big and nef, $c_1(L)$ is primitive in $H^2(S, \mathbb{Z})$ and $L^2 = 2g - 2$. We denote by $\mathcal{F}_g$ the moduli space of these objects. The period map provides an explicit description of $\mathcal{F}_g$ as a period domain:

$$\begin{array}{ccc} & & D_{\Lambda_g} \\ & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{F}_g & \xrightarrow{\mathcal{P}} & D_{\Lambda_g}/\Gamma_g \end{array}$$

where $\Lambda_g = E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \omega \cdot \mathbb{Z}$ with $\omega^2 = 2 - 2g$ and Γ_g is the monodromy group that is the following discrete subgroup

$$\Gamma_g = \{\alpha \in O^+(\Lambda_g) | \alpha \text{ acts trivially on } \Lambda_g^\vee/\Lambda_g\}.$$

The quotient D_{Λ_g}/Γ_g is a smooth quasi-projective variety with at most finite quotient singularities [2].

1.2. **NL-cycles.** There are many different non-equivalent definitions of Noether–Lefschetz cycles on $\mathcal{F}_g$, but most of them span the same subgroup of the rational cohomology groups. For the most part of these notes, we just consider their rational span, so it is sufficient to use a definition without paying attention to the coefficients. A brief overview

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of some of these definitions is given in [2]. For the reader's convenience let us fix the notation that we use in these notes.

Definition 1.1. Let $M \subset \Lambda_g$ be a lattice such that the restriction of the bilinear form on M is negative definite. The *NL-cycles* associated with M is the image of $M^\perp \subset D_{\Lambda_g}$ via the quotient map π.

In the case of codimension-one cycles, instead of considering a negative sublattice we can define NL-divisors as follows. If $M = \langle \beta \rangle$, $\beta^2 = 2h-2$, $\beta \cdot L = d$ and $\langle \beta, c_1(L) \rangle$ is saturated in $H^2(S, \mathbb{Z})$, then we denote the (Weil) NL-divisor $\pi(M^\perp)$ by $D_{h,d}$.

Remark 1.2. *The condition that the subgroup $\langle \beta, c_1(L) \rangle$ is saturated in $H^2(S, \mathbb{Z})$ is needed to ensure that $D_{h,d}$ is well defined.*

Remark 1.3. *The NL-divisors with the above definition are only Weil divisors. Since $\mathcal{F}_g$ has only finite quotient singularities it is $\mathbb{Q}$-factorial. Hence we can naturally consider the Noether–Lefschetz divisors with our definition as $\mathbb{Q}$-Cartier divisors.*

2. Some explicit descriptions

In this section, we present a strategy to give an explicit description of the rational Picard group of $\mathcal{F}_g$ that works for low genus. We focus on describing the rational Picard group of $\mathcal{F}_g$, but this strategy could be helpful for a good understanding of other geometrical properties of $\mathcal{F}_g$.

The results presented in this part are mainly based on [5] and [4], where the interested reader can find the missing details and more accurate descriptions. We should cite also [9], where some interesting related results were proven.

We present the general idea, then we describe with some details the case of K3 surfaces with a polarization of genus 4 and then we sketch the natural adaptation that works up to genus 12.

2.1. **Idea.** The idea is very easy. We construct a moduli space $\mathcal{M}_g$ that parametrizes essentially the same objects of $\mathcal{F}_g$ but has a better geometric description in terms of NL-divisors

$$\begin{array}{ccc} \mathcal{F}_g & \xrightarrow{\mathcal{P}} & D_{\Lambda_g}/\Gamma_g \\ \updownarrow{\scriptstyle f} & & \\ \mathcal{M}_g & & \end{array}$$

and we translate the properties of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_g)$ into properties of $\mathcal{M}_g$ and of f.

2.2. **Case g=4.** In the case $g = 4$, we are parametrizing K3 surfaces with a quasi-polarization of degree 6.

By classical vanishing theorems and the Riemann–Roch theorem, the polarization gives a rational map

$$\phi_L\colon S \dashrightarrow |L| = \mathbb{P}^4.$$

We denote by $S' := \overline{\phi_L(S)}$ the *projective model* of (S, L).

If S is general, *i.e.* $\mathrm{Pic}(S) = \langle L\rangle$, then L is very ample and the projective model S' is a complete intersection of a smooth quadric Q and a smooth cubic C in $\mathbb{P}^4$. We denote by $\mathcal{M}_4$ the moduli space of such complete intersections of smooth sections of degree 2 and 3 in $\mathbb{P}^4$. More concretely, this moduli space can be constructed in the following manner: fix a smooth quadric Q in $\mathbb{P}^4$; this can be done without losing any complete intersection because all smooth quadrics are projectively equivalent. Then, we see C as a section on Q, *i.e.* $C \in |\mathcal{O}_Q(3)| = \mathbb{P}^{29}$. We consider the open subset $\mathcal{U} \subset |\mathcal{O}_Q(3)|$ of cubics that are smooth in $\mathbb{P}^4$ and such that the obtained surface has at most ADE singularities. Note that the complement of $\mathcal{U}$ in $|\mathcal{O}_Q(3)|$ is the discriminant divisor, hence has codimension one.

We can write

$$\mathcal{M}_4 := \mathcal{U}/\operatorname{Aut}(Q) \hookrightarrow \mathcal{F}_4 = \mathcal{M}_4 \sqcup \Delta$$

where Δ is the complement of $\mathcal{M}_4$ inside $\mathcal{F}_4$.

For our purpose, this description is useful because all the line bundles of $\mathcal{F}_4$ can be obtained as products of line bundles on $\mathcal{M}_4$ and the irreducible components of Δ.

Now we describe the components of Δ in terms of Noether–Lefschetz divisors. Essentially we should understand the linear map given by the quasi-polarization in the case the projective model is not a complete intersection of a smooth quadric and a smooth cubic.

Lemma 2.1. *There are the following (non disjoint) possibilities:*

(1) $\mathrm{Bl}(L) \neq \emptyset \iff (S, L) \in D_{1,1}$.

(2) *L is hyperelliptic* $\iff (S, L) \in D_{2,1}$.

If we are not in the previous case, then ϕ_L is birational and $S' = Q \cap C$ for a quadric and a cubic in $\mathbb{P}^4$. Then we have:

(3) *Q is singular* $\iff (S, L) \in D_{3,1}$.

(4) *C is singular* $\iff (S, L) \in D_{4,1}$.

(5) *If we are not in the previous cases then $(S, L) \in \mathcal{M}_4$.*

Proof. Since L is a nef divisor then by [7, Section 2] $\mathrm{Bl}(L) \neq \emptyset$ if and only if we can write $L = 4E + F$ where F is the base locus of L. In this case $\mathrm{Pic}(S) = \langle L, E\rangle$ with $E^2 = 0$ and $E \cdot L = 1$.

If we are not in the first case, we can suppose L is base point free. Hence there are two possibilities: ϕ_L has degree 2 or degree 1. If the degree is 2 this means that (S, L) is hyperelliptic and by [7, Theorem

5.2] this is equivalent to the existence of an irreducible curve E of arithmetic genus one such that $E \cdot L = 2$.

Now, we can suppose the projective model is a degree 6 surface in $\mathbb{P}^4$ that is a complete intersection of a quadric and a cubic.

If the quadric threefold Q contains a curve of degree 3 then it must be singular. Vice versa, if Q is singular, an explicit computation shows that S' contains a genus-one curve E of degree 3.

Similarly one can show the equivalence in (4).

The last equivalence follows. □

Hence we can write $\Delta = \bigcup_{k=1}^{4} D_{k,1}$.

Now, we should understand the Picard group of $\mathcal{M}_4$. We have

$$\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_4) = \mathrm{Pic}_{\mathbb{Q}}(\mathcal{U}/\,\mathrm{SO}(5,\mathbb{C})) \xrightarrow{j} \mathrm{Pic}_{\mathbb{Q}}(\mathcal{U}) = 0$$

where j is an inclusion because $\mathrm{SO}(5,\mathbb{C})$, that is the automorphism group of Q, is a reductive simple group, and $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{U}) = 0$ because $\mathcal{U}$ is the complement of a divisor in a projective space. This proves that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_4)$ is spanned by $D_{k,1}$ for $k = 1, \ldots, 4$.

We conclude with a dimension count: By the results of Maulik and Pandharipande [6], presented by Sertöz in [3], there is an explicit formula for the dimension of the span of Noether–Lefschetz divisors inside the rational Picard group. In our case, this formula tells us that this dimension is 4. This implies that the divisors $D_{k,1}$ for $k = 1, \ldots, 4$ are independent and hence they form a basis of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_4)$.

2.3. **Case** $6 \leq g \leq 10$ **or** $g = 12$**.** For genus $2, \ldots 5$, the same strategy gives similar results. An adaptation of this argument provide a basis of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_4)$. In this paragraph we briefly sketch the steps for $6 \leq g \leq 12$, $g \neq 11$.

(1) A Brill–Noether general[1] (S, L) of genus in our range is birational to a complete intersection with respect to a vector bundle on a homogeneous space. This construction is very explicit and was originally due to Mukai [8].

(2) The moduli space of BN-general (S, L) can be constructed as a GIT quotient $\mathcal{M}_g = \mathcal{U}_g/\mathcal{G}_g$ where $\mathcal{U}_g$ is the complement of the discriminant in the Grassmannian parametrizing our complete intersection, and $\mathcal{G}_g$ is a natural group acting on $\mathcal{U}_g$.

(3) Non BN-general K3 surfaces are contained in some Noether–Lefschetz divisors. More precisely one describe explicitly the NL-divisors of K3 surfaces not parametrized by $\mathcal{M}_g$, *i.e.* $\mathcal{F}_g = \mathcal{M}_g \cup (\bigcup D_{h,d})$.

(4) As in the case of genus 4, $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_g) = 0$. For genus $g = 6, \ldots, 10$ the dimension of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_g)$ is the same as the number

[1] A quasi-polarized K3 surface (S, L) is Brill–Noether general if the inequality $h^0(M) \cdot h^0(N) < h^0(L)$ holds for any pair (M, N) of non-trivial line bundles such that $M \otimes N = L$.

of irreducible components of $\mathcal{F}_g - \mathcal{M}_g$. For $g = 12$ there is one relation, that can be computed explicitly, between the boundary components. Here we are using, as in the case $g = 4$, that the dimension of the span of NL-divisors is known.

3. General results

This section is devoted to a brief discussion about Conjecture 0.1 and Problem 0.3. The results that we present are contained in [1] and in the references therein. In that paper, the authors consider the Shimura variety $\mathcal{F}_g$ from an arithmetic point of view and present the arguments in much more generality, which is not needed for our purposes.

We try to give the idea of their results without giving any technical definition.

As an application of their results, they provide a positive answer to Conjecture 0.1 and the first answer to Problem 0.3.

Theorem 3.1. *For k at most 4 and $g \geq 2$ we have*

$$H^{2k}(\mathcal{F}_g, \mathbb{Q}) = \langle \textit{NL-cycles of codimension } k \rangle_{\mathbb{Q}}.$$

Moreover $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_g) = \langle \textit{NL-divisors} \rangle_{\mathbb{Q}}$.

A natural question that arises looking at this result is the sharpness of the bound.

Question 3.2. *Is the bound $k \leq 4$ of the previous theorem sharp?*

It turns out that the answer to this question is yes [1, end of the proof of Theorem 3.6], since for $k > 4$ the Hodge decomposition of the complex cohomology of $\mathcal{F}_g$ is not of pure type. In order to make a more precise question, we need a couple of remarks.

Remark 3.3. *First of all, $\mathcal{F}_g$ is a singular quasi-projective variety, so one should specify in which cohomology we are considering the Hodge decomposition. There are a lot of different cohomology theories that are used together to study these Shimura varieties, and all of them give isomorphic cohomology groups in the range we are considering. We refer to Debarre's notes for a brief explanation of this point [2, Section 2.4].*

From now on we will use the following notation:

$$Y_q := \Gamma \setminus D_q$$

where $D_q = \mathrm{SO}(2, q)/\,\mathrm{SO}(2) \times \mathrm{SO}(q)$, Γ is a discrete group acting on it. We assume that $q > 3$ and that Y_q is smooth. We omit the dependence of Y_q from Γ to avoid heavy notation.

Remark 3.4. *If we take Γ to be the monodromy group of a polarized K3 surface, then Y_{19} is isomorphic to the moduli space of quasi-polarized K3 surface [2, Section 1].*

Remark 3.5. *The complex cohomology groups of Y_q has a Hodge decomposition*

$$H^k(Y_q, \mathbb{C}) = \bigoplus_{a+b=k} H^{a,b}(Y_q).$$

Question 3.6. *Which is the maximum number k such that $H^{k,k}(Y_q) = \langle NL\text{-cycles of codimension } k\rangle_{\mathbb{C}}$?*

There is the following partial answer to this question.

Theorem 3.7. *If $k < \frac{q+1}{3}$ then*

$$H^{k,k}(Y_q) = \langle NL\text{-cycles of codimension } k\rangle_{\mathbb{C}}.$$

We can deduce Theorem 3.1 from this more general result and by the vanishing of some Hodge groups on Shimura varieties.

Proof of Theorem 3.1. The moduli space D_{Λ_g}/Γ_g is singular as a variety but it is a smooth orbifold *i.e.* there exists a finite index subgroup $\Gamma < \Gamma_g$ such that the variety $Y_{19} := D_{\Lambda_g}/\Gamma$ is smooth and has a natural map that we denote by π to $\mathcal{F}_g$. By a vanishing result due to Zucker and by Theorem 3.7, for $k < q/4 = 19/4$ we have

$$H^{2k}(Y_{19}, \mathbb{C}) = H^{k,k}(Y_{19}) = \langle \text{NL-cycles of codimension } k\rangle_{\mathbb{C}}$$

and hence the (trivial) Hodge decomposition holds over $\mathbb{Q}$. By our definition, the NL-cycles on $\mathcal{F}_g$ are the image of the NL-cycles on Y_{19}. On the other hand, the cohomology of $\mathcal{F}_g$ is the invariant part of the cohomology of Y_{19} under the action of Γ_g:

$$H^{2k}(\mathcal{F}_g, \mathbb{Q}) = H^{2k}(Y_{19}, \mathbb{Q})^{\Gamma_g} =$$
$$= \langle \pi(V) |\ \text{V is a NL-cycle of cod k in } Y_{19}\rangle_{\mathbb{Q}}.$$

The statement on the Picard group follows from the vanishing of the first Betti number of $\mathcal{F}_g$. □

Exactly as in the case of smooth projective varieties, there is a Lefschetz operator $L\colon H^k(Y_q, \mathbb{C}) \to H^{k+2}(Y_q, \mathbb{C})$ induced by the Hodge bundle, i.e. the descent of $\mathcal{O}_{\mathbb{P}(\Lambda_g \otimes \mathbb{C})}(1)$. As in the setting of smooth projective varieties, it induces the primitive cohomology.

Definition 3.8. The *primitive cohomology* is

$$H^{k,k}_{\text{prim}}(Y_q) := \ker(L^{q-2k+1}\colon H^{k,k}(Y_q) \to H^{2q-2k+2}(Y_q, \mathbb{C})).$$

The primitive cohomology has an interpretation via certain irreducible representations of $\mathrm{SO}(2,q)$ that makes it much easier to show that the primitive part of the cohomology is spanned by Noether–Lefschetz cycles.

Theorem 3.9. *With the above notation, if $k < \frac{q+1}{3}$ then*

$$H^{k,k}_{prim}(Y_q) = \langle NL\text{-cycles of codimension } k\rangle_{\mathbb{C}}.$$

Let us conclude by showing (the idea of) how to deduce Theorem 3.7 from this result.

Proof of Theorem 3.7. Starting from the decomposition in the primitive parts of the cohomology groups of Y_q

$$H^{k,k}(Y_q) = \bigoplus_{t=0}^{k} L^{k-t} H^{t,t}_{\text{prim}}(Y_q)$$

there are two further non-trivial steps for this proof:

- The Lefschetz operator sends classes spanned by NL-cycles into classes spanned by NL-cycles.
- If α and β are spanned by NL-cycles, then also their product is.

□

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Mathematisches Institut, Bonn, Germany
Email address: anella@math.uni-bonn.de