# Moduli Spaces of HK Manifolds

**Def** $X$ compact Kähler complex manifold $\dim_{\mathbb{C}} X = 2n$

Say $X$ HK if

- $X$ simply connected
- $H^0(X, \Omega_X^2) = \mathbb{C}\cdot\eta$ non-degenerate sympl. form

**Fact**: $\exists$ integral, primitive quadratic form $q_X$ on $H^2(X,\mathbb{Z})$ s.t.

- $q_X$ non-degenerate of sgn $(3, b_2(X)-3)$
- Hodge decomp. orthogonal wrt $q_X$
- $q_X$ deformation invariant

Beauville-Bogomolov-Fujiki form

$$q_X(\alpha)^n = c_X \cdot \int_X \alpha^{2n}, \quad \forall \alpha \in H^2(X,\mathbb{Z})$$

Fujiki const.

## 1. The moduli space of marked HKs

Fix $(\Lambda, q)$ non-deg. lattice, $rk(\Lambda) = b \geq 3$, $sgn(\Lambda) = (3, b-3)$

**Def** $(X, \varphi)$ marked HK if

- $X$ HK
- $\varphi : (H^2(X,\mathbb{Z}), q_X) \xrightarrow{\sim} (\Lambda, q)$

Can speak about families of marked HKs, isomorphisms, over analytic base space in the natural way

$\rightsquigarrow$ get a functor for families of marked HKs.

K3 HK univ. family

Thm [Beauville, Huybrechts, Markman]
There exists a coarse moduli space $\mathfrak{M}_\Lambda$ param. marked HKs, which is smooth, non-Hausdorff complex space, $\dim(\mathfrak{M}_\Lambda) = b-2$, together w/ universal family

unique up to isom's

$$\pi : \mathcal{X}_\Lambda \to \mathfrak{M}_\Lambda$$
$$\Phi : R^2\pi_*\mathbb{Z} \xrightarrow{\sim} \underline{\Lambda}$$

■

Rmk $X$ HK

$$\mathrm{Aut}_0(X) := \ker\left(\mathrm{Aut}(X) \to O(H^2(X,\mathbb{Z}))\right)$$

Idea: • specializ. argument
• birat'l morph. preserving Kähler class is autom.

Facts: • [Kaledin-Verbitsky] [Hassett-Tschinkel] $\mathrm{Aut}_0(X)$ deformation invariant

$\pi : \mathcal{X} \to B$
$\exists\ \mathcal{A}ut_0(\pi)$ local system / $B$ w/ fiber $\mathrm{Aut}_0(X_b)$ over $b \in B$

• [Huybrechts] $\mathrm{Aut}_0(X)$ finite group

Idea: • $f \in \mathrm{Aut}_0(X)$ preserves HK metric $\rightsquigarrow$ isom. $\rightsquigarrow$ compact gp
• $H^0(X, T_X) = 0 \rightsquigarrow$ discrete gp

$\mathfrak{M}_\Lambda^\circ \subseteq \mathfrak{M}_\Lambda$ conn. comp. w/ $\mathrm{Aut}_0$ trivial

$\rightsquigarrow \mathfrak{M}_\Lambda^\circ$ fine moduli space.

▲

Exs (1) [Burns-Rapoport] $X$ K3 surface $\rightsquigarrow$ $\mathrm{Aut}_0(X) = \{id\}$

Idea Pf: $h \in \mathrm{Aut}_0(X)$, $h \neq id$

$\rightsquigarrow$ $h^m = id$ & $h$ symplectic $\leftarrow$ $h^*\eta = \eta$

$\rightsquigarrow$ local coord's

- $\mathrm{Fix}(h)$ = finitely many pts.
- $\forall x \in \mathrm{Fix}(h)$, $d_x h = \begin{pmatrix} \lambda_x & 0 \\ 0 & \lambda_x^{-1} \end{pmatrix}$. $\lambda_x^m = 1$, $\lambda_x \neq 1$.

Holo Lefschetz:

$$2 = \sum_{i=0}^{2} (-1)^i \mathrm{tr}\left(h^*|_{H^i(\mathcal{O}_X)}\right) = \sum_{x \in \mathrm{Fix}(h)} \det(id - d_x h)^{-1} = \sum_{x \in \mathrm{Fix}(h)} \frac{1}{(1-\lambda_x)(1-\lambda_x^{-1})} \leq \frac{|\mathrm{Fix}(h)|}{4}$$

$\rightsquigarrow$ $|\mathrm{Fix}(h)| \leq 8$

Topol. Lefschetz:

$$24 = \sum_{i=0}^{4} (-1)^i \mathrm{tr}\left(h^*|_{H^i(X,\mathbb{Z})}\right) = |\mathrm{Fix}(h)| \leq 8 \quad \text{⚡}$$

(2) [Beauville] $X$ $K3^{[n]}$-type $\rightsquigarrow$ $\mathrm{Aut}_0(X) = \{id\}$

eg., $S$ general w/ $\rho(S) = 0$

Idea Pf: Deform $X$ to $S^{[n]}$, $S$ K3 surface (w/ $\mathrm{Aut}(S) = \{id\}$)

$\rightsquigarrow$ all $h \in \mathrm{Aut}_0(S^{[n]})$ respect exc. div. $\rightsquigarrow$ come from $\mathrm{Aut}(S)$.

(3) [Beauville] [Boissière-Nieper-Wißkirchen-Sarti] $X$ $\mathrm{Kum}_n$-type $\rightsquigarrow$ $\mathrm{Aut}_0(X) \cong \left(\mathbb{Z}/(n+1)\mathbb{Z}\right)^4 \rtimes \mathbb{Z}/2\mathbb{Z}$

Idea Pf: Deform $X$ to $\mathrm{Kum}_n(A)$, $A$ abelian surface

Then we have an inclusion

$$A[n+1] \rtimes \mathrm{Aut}(A) \longrightarrow \mathrm{Aut}(\mathrm{Kum}_n(A))$$

transl. by $(n+1)$-torsion pts (all act trivially on $H^2(\mathrm{Kum}_n(A), \mathbb{Z})$)

the trivial action on $H^2(\mathrm{Kum}_n(A), \mathbb{Z})$ comes from $-id_A$

Rmk
- [Oguiso] $X$ $\mathrm{Kum}_n$-type $\rightsquigarrow$ $\mathrm{Aut}(X) \hookrightarrow Gl(H^*(X,\mathbb{Z})_{\text{tf}})$
- [Kapfer-Menet] $X = \mathrm{Kum}_2(A) \rightsquigarrow -\mathrm{id}_A$ acts non-trivially on both $H^3(X,\mathbb{Z})$ & $H^4(X,\mathbb{Z})$ (both torsion-free in this case)

in general ?

(4) [Mongardi-Wandel]
- $X$ OG10-type $\rightsquigarrow$ $\mathrm{Aut}_0(X) = \{\mathrm{id}\}$

reduce to K3

- $X$ OG6-type $\rightsquigarrow$ $\mathrm{Aut}_0(X) \cong (\mathbb{Z}/2\mathbb{Z})^8$

reduce to $A$ ab. surface
- $-\mathrm{id}_A$ acts trivially
- $(A\times\hat{A})[2]$ acts non-triv. but trivially on $H^2$

▲

## Idea Pf Thm

St.1 : Local case

$X$ HK $\underset{\substack{H^0(X,T_X)=0\\ \text{[Kuranishi]}}}{\Longrightarrow}$ $\exists\, \mathcal{X} \to \mathrm{Def}(X)$ universal deformation

$\underset{\substack{\omega_X \cong \mathcal{O}_X\\ \text{[Bogomolov, Tian, Todorov]}\\ \text{[Ran, Kawamata]}}}{\Longrightarrow}$ $(\mathrm{Def}(X),0)$ smooth w/ $T_0\mathrm{Def}(X) = H^1(X,T_X)$

St.2 : Local period map

$\Lambda$ lattice, $\mathrm{sgn}(\Lambda) = (3, b-3)$

$Q_\Lambda := \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) : q(x)=0,\ q(x,\bar{x})>0\}$

$\overset{\sim}{\underset{\text{diff.}}{\to}} O(\Lambda_{\mathbb{R}})/(SO(2)\times O(1,b-3)) \cong Gr_2^{po}(\Lambda_{\mathbb{R}})$

($\rightsquigarrow$ simply connected & smooth)

$$\mathcal{P}_X : \mathrm{Def}(X) \longrightarrow Q_{H^2(X,\mathbb{Z})}$$
$$t \longmapsto [\eta_t]$$

Thm $X$ HK $\Rightarrow$ $\mathcal{P}_X$ local isom. at 0

$\Rightarrow$ injective

Idea: $d\mathcal{P}_{X_0} : H^1(X,T) \longrightarrow \mathrm{Hom}(\underbrace{H^{2,0}(X)}_{\cong\, \mathbb{C}\eta}, H^2(X,\mathbb{C})/H^{2,0}(X))$

$v \longmapsto (\eta \mapsto \eta \lrcorner v \in H^{1,1}(X))$

St.3: Gluing via marking

$(X,\varphi)$ marked HK

$\rightsquigarrow \quad \varphi \circ P_X : \mathrm{Def}(X) \longrightarrow Q_{H^2(X,\mathbb{Z})} \xrightarrow{\sim} Q_\Lambda$

$\underset{\text{St.2}}{\rightsquigarrow}$ can glue $\mathrm{Def}(X)$ to $\mathfrak{M}_\Lambda$ smooth (non-Hausdorff) and

global period map

$$P : \mathfrak{M}_\Lambda \longrightarrow Q_\Lambda$$
$$(X,\varphi) \longmapsto \varphi(H^{2,0}(X))$$

$\underset{\text{St.1}}{\rightsquigarrow}$ $\mathfrak{M}_\Lambda$ coarse moduli space, $\dim \mathfrak{M}_\Lambda = b-2$

St.4: Universal family, if $\mathrm{Aut}_0$ = trivial

$\mathfrak{M}_\Lambda^\circ \subseteq \mathfrak{M}_\Lambda$ conn. cpt. w/ $\mathrm{Aut}_0$ trivial

$\rightsquigarrow$ can glue the family $\mathcal{X}$ as well (uniquely!)

$\rightsquigarrow$ univ. family $\pi : \mathcal{X}_\Lambda^\circ \to \mathfrak{M}_\Lambda^\circ$

$\rightsquigarrow$ $\mathfrak{M}_\Lambda^\circ$ fine moduli space.

St. 5 : Universal family, if $Aut_0$ not trivial ⟵ see also [Looijenga]

Recall: Global Torelli Thm [Verbitsky, Huybrechts, Markman]

$\mathfrak{M}^\circ_\Lambda \subseteq \mathfrak{M}_\Lambda$ conn. cpt.

Then

$$P|_{\mathfrak{M}^\circ_\Lambda} : \mathfrak{M}^\circ_\Lambda \to \mathcal{Q}_\Lambda \quad \text{generically injective \& surjective}$$

More precis.,

$$P(X,\varphi) = P(X',\varphi') \iff (X,\varphi) \text{ and } (X',\varphi') \text{ inseparable pts}$$

$$(\Rightarrow X \text{ and } X' \text{ bimeromorphic})$$

■

(5.1) Fix $G$ finite gp. $\rightsquigarrow$ $\mathfrak{M}_{\Lambda,G}$ moduli space param.

$(X,\varphi,\psi)$ where
- $(X,\varphi)$ marked HK
- $\psi : Aut_0(X) \xrightarrow{\sim} G$

Rmk $Aut(X,\varphi,\psi) \cong Z(G)$ ⟵ the center of $G$

Then:
- forg: $\mathfrak{M}_{\Lambda,G} \to \mathfrak{M}_\Lambda$ restrict to isom on each connected cpt.

these are standard, except need to be careful since $\mathfrak{M}^\circ_\Lambda$ not Hausdorff (!)

- $\exists$ univ. family over $\mathfrak{M}^\circ_{\Lambda,G}$ iff the "associated class"

$$[\underline{\mathfrak{M}}_{\Lambda,G}] \in \check{H}^2(\mathfrak{M}^\circ_{\Lambda,G}, Z(G))$$

vanishes

Key pts:
- $\mathcal{Q}_\Lambda$ simply conn. $\Rightarrow$ every local system on $\mathfrak{M}^\circ_\Lambda$ is trivial
- $\underline{\mathfrak{M}}_{\Lambda,G}$ associated gerbe is actually classified by $H^2$ w/ band $Z(G)$

(5.2) <u>Twistor family</u>

$X$ HK, $\omega$ Kähler class $\rightsquigarrow$ $(M, I, g)$

underlying manifold — complex structure — Ricci flat hermitian metric (uniquely assoc. to $\omega$)

Vary $I$, by fixing $(M, g)$

$\rightsquigarrow$ $Tw_\omega \xrightarrow{\pi} \mathbb{P}^1_\omega$ family of HK

$(X, \varphi)$ marked HK $\rightsquigarrow$ $Tw_\omega$ has natural marking as well

$$(X,\varphi) \in \mathfrak{M}^\circ_\Lambda \rightsquigarrow \begin{tikzcd} \mathbb{P}^1_\omega \arrow[r, hook, "i_\omega"] \arrow[dr, "j_\omega"'] & \mathfrak{M}^\circ_\Lambda \arrow[d] \\ & Q_\Lambda \end{tikzcd}$$

conic

$\mathbb{P}(\langle \eta_I, \eta_J, \eta_K \rangle) \cap Q_\Lambda$

(5.3) <u>Key Lemma</u> $(X, \varphi) \in \mathfrak{M}^\circ_\Lambda$, $\omega$ Kähler form on $X$

Then

- $\forall A$ ab. gp., $\check{H}^*(\mathfrak{M}^\circ_\Lambda, A) \overset{\text{Torelli}}{\cong} \check{H}^*(Q_\Lambda, A) \cong \check{H}^*(\mathbb{P}^1_\omega, A)$ — $j_\omega$ homotopy equiv.

  $i^*_\omega$

- $i^*_\omega [\underline{\mathfrak{M}}_{\Lambda, G}] = 0$

  because a family exists over $\mathbb{P}^1_\omega$, namely $Tw_\omega$ (!)

$\blacktriangle$

$\rightsquigarrow$ Thm is proved !

$\blacksquare$

<u>Rmk</u> Fix polarization type $\underline{h} = O(\Lambda)\cdot h \subseteq \Lambda$, $h \in \Lambda$ primitive, $q(h)>0$.

We can look at moduli space of polarized marked pairs of type $\underline{h}$

$\rightsquigarrow \quad \mathfrak{M}_{\Lambda,\underline{h}}$

$\underset{\text{(Olivier's talk)}}{\rightsquigarrow} \quad F_{\Lambda,h} \cong \mathfrak{M}_{\Lambda,\underline{h}}/O(\Lambda)$

(to get the corresponding period domain and period map, we'll introduce the monodromy group next)

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see also [Looijenga] 

## 2. The monodromy group

**Def** X HK

- $\mathrm{Mon}(X) \subseteq Gl(H^*(X,\mathbb{Z}))$ <u>monodromy group</u>

  subgp. gen. by images of monodromy represents

- $\mathrm{Mon}^2(X) \subseteq Gl(H^2(X,\mathbb{Z}))$

  degree-2 part

<u>Facts</u> [Verbitsky] see also Erratum on arXiv

preserve cpt. of positive cone cont. Kähler cone

- $\mathrm{Mon}^2(X) \subseteq O^+(H^2(X,\mathbb{Z}))$ <u>finite index subgp.</u> normal?

- $\exists$ exact seq. of gps

$$\mathrm{Aut}_0(X) \longrightarrow \mathrm{Mon}(X) \longrightarrow \mathrm{Mon}^2(X) \longrightarrow 1$$

<u>Global Torelli Thm</u>, cohomological version

$\varphi \in \mathrm{Mon}^2(X)$ st.
- $\varphi$ Hodge isom.
- $\varphi$(Kähler class) is Kähler class

Then $\exists f \in \mathrm{Aut}(X)$ st. $\varphi = f_*$

■

$$\rightsquigarrow \quad F^0_{\Lambda,h} \xrightarrow{P} \mathrm{Mon}^0(\Lambda,h) \backslash \Omega_{\Lambda,h}$$

<u>open immersion</u> (Olivier's talk)

$\Omega_{\Lambda,h} = \{ x \in \mathbb{P}(h^{\perp}_{\mathbb{C}}) : q(x)=0,\ q(x,\bar{x})>0 \}$

polarized monodromy gp. (stabilizer of $F_1(H)$) (it depends, a priori, on the choice of conn. cpt. $F^0_{\Lambda,h}$)

<u>Verbitsky's definition</u>:

X HK $\longleftrightarrow$ $(M, I)$

M: underlying manifold; I: cmplx str.

- $\Gamma_M := \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ <u>mapping class group</u>

- $\mathrm{Teich}_M := \mathrm{Cmplx}(M)/\mathrm{Diff}_0(M)$ <u>Teichmüller space</u>

- $\mathrm{Teich}^0_M \subseteq \mathrm{Teich}_M$ conn. cpt. containing I

- $\Gamma^0_M \subseteq \Gamma_M$ sgp. preserving $\mathrm{Teich}^0_M$

$\rightsquigarrow \mathrm{Mon}(X) = \mathrm{im}(\Gamma^0_M \to Gl(H^*(M,\mathbb{Z})))$

uses stronger form of Global Torelli: $\boxed{\mathrm{Teich}^0_M \cong \mathfrak{M}^0_\Lambda}$

<u>Rmk</u>
- the fact that $\mathrm{im}(\Gamma^0_M \to O(H^2(M,\mathbb{Z})))$ has finite index is an ingr. of pf.; it was incorrect in the original version, but fixed in the Erratum.
- if $f \in \Gamma^0_M$ acts trivially on $H^2(M,\mathbb{Z})$, then by Torelli, it must fix I $\rightsquigarrow$ it comes from holo autom. of $(M, I)$ ▲

<u>Rmk</u> • Can also define moduli spaces of $h$-polarized HK w/ (<u>full</u>) <u>$\ell$-level</u> structure $\mathfrak{F}^0_{\Lambda,h,\ell} \to \mathfrak{F}^0_{\Lambda,h}$ <u>finite étale</u>

(by taking kernel of $\Gamma^0_{M,h} \to GL(H^2_{prim}(M,\mathbb{Z}/\ell\mathbb{Z}))$)

(or in $O(H^2_{prim}(M,\mathbb{Z}/\ell\mathbb{Z}))$) equivalently, by further fixing $H^2(X,\mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{\sim} H^2(M,\mathbb{Z})\otimes \mathbb{Z}/\ell\mathbb{Z}$

Again, the appropriate period map is <u>open embedding</u>

• Can also define moduli spaces of <u>lattice-polarized</u> ($h$-polarized) HK

[Dolgachev]
[Camere]

$$\Sigma \underset{\text{prim. emb.}}{\hookrightarrow} \Lambda, \quad sgn(\Sigma) = (1, r)$$

$$(X,\phi), \quad \phi : \Sigma \hookrightarrow NS(X)$$

see Olivier's notes

<u>Exs</u> (1) $X$ K3 surface $\rightsquigarrow$ $Mon^2(X) = O^+(H^2(X,\mathbb{Z}))$

(2) [Markman] $X$ $K3^{[m]}$-type $\rightsquigarrow$ $Mon^2(X) = \langle \rho_u : u \in \Lambda, u^2 = \pm 2 \rangle$ refl. at $u$

$= \{\varphi \in O^+(\Lambda) : \varphi \text{ acts as } \pm 1 \text{ on } D(\Lambda)\}$

$\Lambda = H^2(X,\mathbb{Z}) = U^3 \oplus E_8(-1)^2 \oplus \langle 2-2m \rangle$

$D(\Lambda) = \Lambda^\vee/\Lambda \simeq \mathbb{Z}/(2m-2)\mathbb{Z}$

<u>Idea</u>: Reduce to $X = M_v(\sigma)$ on $S$ K3 surf. and use derived categories

(3) [Markman-Mehrotra] [Mongardi] [Markman] $X$ $Kum_n$-type $\rightsquigarrow$ $Mon^2(X) = \left\{ \varphi \in O^+(\Lambda) : \begin{array}{l} \bullet\ \varphi \text{ acts as } \boxed{\pm 1} \text{ on } D(\Lambda) \\ \bullet\ \det(\varphi)\cdot \chi(\varphi) = 1 \end{array} \right\}$ $=: \chi(\varphi)$

$\Lambda = U^3 \oplus \langle -2-2m \rangle$

$D(\Lambda) \simeq \mathbb{Z}/(2m+2)\mathbb{Z}$

<u>Idea</u>: $X = Kum_n(A)$, $A$ abelian surface

$S^+_X = H^{ev}(A,\mathbb{Z})$ $(\simeq U^4)$

$v = ch(\mathcal{I}_n) = (1, 0, -n-1)$

and $Mon(X) \simeq \widetilde{G}(S^+_X)^{ev}_v$

$$1 \to A[n+1] \to \widetilde{G}(S^+_X)^{ev}_v \to G(S^+_X)^{ev}_v \to 1$$

$$1 \to Spin(S^+)_v \to G(S^+_X)^{ev}_v \to \mathbb{Z}/2\mathbb{Z} \to 1$$

$$\downarrow 2\text{-}1$$

$$SO^+(\Lambda)$$

think in terms of derived categories

(4) [Onorati] $X$ OG10-type $\rightsquigarrow$ $\mathrm{Mon}^2(X) = O^+(\Lambda)$

$\Lambda = U^3 \oplus E_8(-1)^2 \oplus A_2(-1)$

(5) [Mongardi-Rapagnetta] $X$ OG6-type $\rightsquigarrow$ $\mathrm{Mon}^2(X) = O^+(\Lambda)$

$\Lambda = U^3 \oplus \langle -2 \rangle \oplus \langle -2 \rangle$

$\mathrm{Mon}(X) = ?$

$\blacktriangle$

. .

## 3. Group actions on cohomology

$X$ HK, $\dim_{\mathbb{C}} X = 2n$

$\Lambda := H^2(X, \mathbb{Z})$

(I) Monodromy action

[Markman]

algebraic gp. $/\mathbb{Q}$

$G_X := SO(\Lambda_{\mathbb{Q}})$

$\rightsquigarrow$ (Mon$^2(X)$ has finite index) $\quad G_X \subseteq \overline{\mathrm{Mon}^2(X)}$

$\widetilde{G}_X := \mathrm{Spin}(\Lambda_{\mathbb{Q}})$

$\rightsquigarrow$ (finite kernel) $\quad \widetilde{G}_X \to \overline{\mathrm{Mon}(X)}$

$\rightsquigarrow \quad \widetilde{G}_X$ acts on $H^*(X, \mathbb{Q})$

(compatibly w/ cup product, respecting Hodge structure, and extending natural action on $H^2(X,\mathbb{Q})$ )

**Exs** (1) $X$ $K3^{[n]}$-type or OG10-type $\rightsquigarrow$ $\mathrm{Mon}(X) = \mathrm{Mon}^2(X)$

$\rightsquigarrow$ action of $\widetilde{G}_X$ factors thr. $G_X$.

(2) $X$ $\mathrm{Kum}_n$-type $\underset{(\text{Ex } 2.3)}{\rightsquigarrow}$ $\widetilde{G}_X$ acts via $\mathrm{Spin}(S^+)_v$ and

thus does not factor thr. $G_X$

(this seems in contradiction w/ [Bergeron-Li, §2.4] ? )
(Mirko's question)

▲

## (II) LLV action

[Verbitsky]
[Looijenga-Lunts]

$R := \mathrm{End}(H^*(X,\mathbb{R}))$

- $h \in R$ acting as $(i-2n)$ on $H^i(X,\mathbb{R})$
- $a \in H^2(X,\mathbb{R}) \rightsquigarrow e_a \in R$ acting by multipl. by $a$

unique!

**Def** $a \in H^2(X,\mathbb{R})$ Lefschetz if $\exists\, f_a \in R$ st.

$$[e_a, f_a] = h \quad , \quad [h, e_a] = 2e_a \quad , \quad [h, f_a] = 2f_a$$

Say $(e_a, h, f_a)$ Lefschetz triple

ex $a \in H^{1,1}(X)$ Kähler class
$\rightsquigarrow e_a = L_a$ Lefschetz oper.
$f_a = \Lambda_a$ dual Lefschetz oper.

▲

**Def** $\mathfrak{g}_X :=$ Lie alg. generated by all Lefschetz triples

defined over $\mathbb{Q}$

**Facts:**
- $\mathfrak{g}_X \cong \mathfrak{so}(H^2(X,\mathbb{R}) \oplus U_{\mathbb{R}})$
- $\mathfrak{g}_{X,0}^{ss} \cong \mathfrak{so}(H^2(X,\mathbb{R}))$ integrates to an action of $\mathrm{Spin}(H^2(X,\mathbb{R}))$ on $H^*(X,\mathbb{R})$ (compatible w/ cup product and extending natural action on $H^2(X,\mathbb{R})$)

  semisimple part of degree 0

- On even cohom., it factors via action of $G_X$ on $H^{ev}(X,\mathbb{Q})$.

Exs [Markman]

(1) $X$ $K3^{[n]}$-type $\rightsquigarrow$ 2 actions

$$\rho_{Mon} : G_X \longrightarrow Gl(H^*(X,\mathbb{Q}))$$

$$\rho_{LLV} : G_X \longrightarrow Gl(H^*(X,\mathbb{Q}))$$

and so on a possibly index 2 subgp. (it is index 2, if $n \geq 3$)

extending natural actions on $H^2(X,\mathbb{Q})$

They coincide on $Mon^2(X) \cap SO(H^2(X,\mathbb{Z}))$ up to a sign

(2) $X$ $Kum_n$-type $\rightsquigarrow$ the 2 actions coincide on

$H^*(X,\mathbb{C})^{\Gamma_X}$ for the gp. $Spin(H^2(X,\mathbb{C}))$

$\Gamma_X \cong (\mathbb{Z}/(n+1)\mathbb{Z})^4 \subseteq Aut_0(X)$

this also seems in contradiction w/ [Bergeron-Li, §2.4.1] ?

▲

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