NOETHER–LEFSCHETZ CYCLES

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Contents

1. Period domains

1.1. **Period domains and their arithmetic quotients.** The general setup is the following. Let Λ be a lattice with signature $(2, s)$, with $s > 0$, and consider the *period domain*[1]

$$\begin{aligned} D_\Lambda &:= \{[\omega] \in \mathbf{P}(\Lambda_{\mathbf{C}}) \mid \omega^2 = 0,\ \omega \cdot \bar\omega > 0\} \\ &= \{\text{oriented positive 2-planes in } \Lambda_{\mathbf{R}}\} \\ &= \mathrm{SO}(2, s)/\,\mathrm{SO}(2) \times \mathrm{SO}(s) \end{aligned}$$

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[1]It is denoted by $\widehat{D}$ in [BLMM, Section 2.3].

where the bijection between the first and the second lines is given by sending $[\omega]$ to the real oriented positive 2-plane spanned by $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$. It has two isomorphic components (corresponding to the two components of the group $\mathrm{SO}(2,s)$) which are Hermitian symmetric domains of dimension s. The stabilizer of each compoonent is $\mathrm{SO}^+(2,s)$ (see Appendix A).

Any subgroup Γ of finite index of the isometry group $\mathrm{O}(\Lambda)$ (an *arithmetic group*) acts properly and discontinuously on D_Λ and, by the Baily–Borel theory, the quotient D_Λ/Γ is a quasi-projective algebraic variety of dimension s with a canonical (projective) compactification $(D_\Lambda/\Gamma)^{BB}$ obtained by adding boundary components of dimensions at most 1. The line bundle $\mathscr{O}(1)$ on $D_\Lambda \subset \mathbf{P}(\Lambda_{\mathbf{C}})$ descends to a line bundle on D_Λ/Γ which is ample; it is the dual of the *Hodge line bundle*.

On the arithmetic side, D_Λ/Γ is the complexification of an *orthogonal Shimura variety* which is defined over $\mathbf{Q}$ ([B, § 1.3 and 1.4]).

We let $\mathrm{O}^+(\Lambda)$ be the stabilizer of either component of D_Λ. It is a subgroup of index 2 of $\mathrm{O}(\Lambda)$ which is the kernel of the spinor norm (see Appendix A). For an element of $\mathrm{SO}(\Lambda)$, being in $\mathrm{SO}^+(\Lambda)$ means preserving the orientation on one (hence any) positive 2-plane.

1.2. **Motivation: hyperkähler varieties.** Why are algebraic geometers interested in the material presented so far? It is because these quotients D_Λ/Γ appear as target spaces of period maps for certain algebraic varieties.

This is the case for polarized hyperkähler varieties. These are pairs (X,H), where H is a primitive ample line bundle on the (smooth projective) hyperkähler variety X. They admit a moduli space/stack, each component of which is quasi-projective.

1.2.1. *Moduli spaces.* The free abelian group $H^2(X,\mathbf{Z})$ is endowed with canonical quadratic form q_X (the Beauville–Bogomolov–Fujiki form) which makes it into a lattice with signature $(3, b_2(X)-3)$. We will consider hyperkähler varieties of a fixed deformation type. All their lattices $(H^2(X,\mathbf{Z}),q_X)$ are then isomorphic to the same lattice Λ. We pick a primitive class $h\in\Lambda$ with $h^2>0$ and consider pairs (X,H) as above for which there is an isometry $(H^2(X,\mathbf{Z}),q_X)\simeq\Lambda$ that takes H to h. The corresponding moduli stack is then a smooth Deligne–Mumford stack that can be coarsely represented by a quasi-projective variety $\mathscr{F}_{\Lambda,h}$.[2] It only depends on the $\mathrm{O}(\Lambda)$-orbit T of h (called a *polarization type*) and may have several connected components.

1.2.2. *Period maps.* Via the isometry $(H^2(X,\mathbf{Z}),q_X)\simeq\Lambda$, the period of X, that is, the point $[H^{2,0}(X)]$ of $\mathbf{P}(H^2(X,\mathbf{C})$, lands into

$$D_{\Lambda,h} := D_{h^\perp} = \{[\omega]\in D_\Lambda \mid \omega\cdot h = 0\}, \tag{1}$$

a lattice that only depends (up to isomorphism) on the polarization type T. The lattice $h^\perp$ has signature $(2, b_2(X)-3)$ and the discussion of Section 1.1 applies: we obtain an *algebraic*[3] period map

$$\mathscr{F}_{\Lambda,T} \longrightarrow D_{\Lambda,h}/\,\mathrm{O}(\Lambda,h),$$

where $\mathrm{O}(\Lambda,h)\subset\mathrm{O}(h^\perp)$ is the image of the injective restriction morphism

$$\{g\in\mathrm{O}(\Lambda)\mid g(h)=h\}\hookrightarrow\mathrm{O}(h^\perp),\quad g\longmapsto g|_{h^\perp}.$$

[2]It is explained in [B, § 1.2] that for K3 surfaces, $\mathscr{F}_{\Lambda,h}$ is defined over $\mathbf{Q}$ (and even over $\mathbf{Z}[\frac12]$). I do not know whether this still holds for general hyperkähler varieties.

[3]For K3 surfaces, it is defined over $\mathbf{Q}$, and even over $\mathbf{Z}[\frac12]$ (see [B, § 1.5]).

Remark 1.1. In all known cases (see Appendix B), the lattice Λ is even and has the form

$$\text{(unimodular lattice)} \oplus U^{\oplus 3} \oplus L, \tag{2}$$

where U is the (unimodular) hyperbolic plane and L has rank 1 or 2. A theorem of Eichler then says that the $\widetilde{\mathrm{O}}(\Lambda)$-orbit of a primitive element $h \in \Lambda$ only depends on h^2 and the element $h_* := h/\operatorname{div}(h)$ of $D(\Lambda) = D(L)$, where $\operatorname{div}(h)$ (the *divisibility* of h) is the positive generator of the subgroup $h \cdot \Lambda$ of $\mathbf{Z}$.

Moreover, the quadratic form of Λ induces a $\mathbf{Q}/2\mathbf{Z}$-valued quadratic form on $D(\Lambda)$ and the map $\pi\colon \mathrm{O}(\Lambda) \longrightarrow \mathrm{O}(D(\Lambda)) = \mathrm{O}(D(L))$ considered earlier is surjective ([N, Remark 1.14.3]). So the polarization types are determined by h^2 (the *degree*) and the $\mathrm{O}(D(\Lambda))$-orbits in $D(\Lambda)$. For K3 surfaces, Λ is unimodular and polarization types are determined by their (even positive) degree. For hyperkähler varieties of $K3^{[m]}$-type, where $m-1$ is a power of a prime, there are m distinct polarization types of a given degree.

1.2.3. *Monodromy groups.* Markman defined the *monodromy group* of X as the subgroup

$$\operatorname{Mon}(X) \subset \mathrm{O}(H^2(X,\mathbf{Z}), q_X) \tag{3}$$

generated by all monodromies for all families of deformations of X. Note that it contains all automorphisms of all deformations of X (take for the base of the deformation a nodal curve). Verbitsky proved that it has finite index and Markman that it is contained in $\mathrm{O}^+(H^2(X,\mathbf{Z}), q_X)$. One also defines

$$\operatorname{Mon}(X,H) := \operatorname{Mon}(X) \cap \mathrm{O}(H^2(X,\mathbf{Z}), H)$$

as the subgroup that stabilizes the polarization H.

If one picks a connected component $\mathscr{F}^0_{\Lambda,h}$ of the moduli space $\mathscr{F}_{\Lambda,h}$, the groups $\operatorname{Mon}(X)$, when $[X]$ describes the set of hyperkähler varieties parametrized by $\mathscr{F}^0_{\Lambda,h}$, can be identified with a subgroup $\operatorname{Mon}^0(\Lambda)$ of $\mathrm{O}(\Lambda)$ which may depend on the choice of the component. The Torelli theorem, proved by Verbitsky, then states that the period maps induces an open embedding ([M2, Lemma 8.1])

$$\mathscr{F}^0_{\Lambda,h} \hookrightarrow D_{\Lambda,h}/\operatorname{Mon}^0(\Lambda,h). \tag{4}$$

When $\operatorname{Mon}(X)$ is normal in $\mathrm{O}(H^2(X,\mathbf{Z}), q_X)$ (which happens in all known cases, see Appendix B), the group $\operatorname{Mon}^0(\Lambda)$ is independent of the choices and we write it as $\operatorname{Mon}(\Lambda)$.

2. Noether–Lefschetz cycles

2.1. **Noether–Lefschetz divisors.** Let (X,H) be a polarized hyperkähler variety with period $[\omega_X] \in \mathbf{P}(H^2(X,\mathbf{C}))$. By the Lefschetz (1,1)-theorem, its Picard group is isomorphic to $H^2(X,\mathbf{Z}) \cap \omega_X^\perp$. It has signature $(1,\rho-1)$. The Noether–Lefschetz locus is the locus (in the moduli space $\mathscr{F}_{\Lambda,h}$) where the rank ρ of this group is at least 2.

In the period domain $D_{\Lambda,h}$, this can be translated into purely algebraic terms: for any $x \in h^\perp$, set

$$D_x := D_{\Lambda,h} \cap x^\perp. \tag{5}$$

When $x^2 < 0$, it is an irreducible hypersurface, which is nothing else than the period domain $D_{h^\perp \cap x^\perp}$. The Noether–Lefschetz locus in $D_{\Lambda,h}$ is then the countable union $\bigcup_{x\in\Lambda,\ x^2<0} D_x$.

Let $\Gamma \subset \mathrm{O}(\Lambda, h)$ be an arithmetic subgroup. The image in the Shimura variety $Y_\Gamma := D_{\Lambda,h}/\Gamma$ of this locus is then

$$\bigcup_{x\in\Lambda,\ x^2<0} D_x/\Gamma_x$$

where Γ_x is the stabilizer of D_x in Γ. This is a countable union of hypersurfaces which we call *Noether–Lefschetz divisors.*

When $\Gamma = \mathrm{Mon}^0(\Lambda, h)$, the trace on $\mathscr{F}^0_{\Lambda,h}$ (via the period map (4)) is the Noether–Lefschetz locus as defined above.

One of the main results of [BLMM] is that when $s \geq 3$, Noether–Lefschetz divisors generate (over $\mathbf{Q}$) the Picard group of the Shimura variety Y_Γ (and of $\mathscr{F}_{\Lambda,h}$).

2.2. **Noether–Lefschetz cycles.** One can generalize the above discussion to r-cycles. Geometrically, this corresponding to looking at hyperkähler varieties with Picard number $\geq r+1$.

Arithmetically, this corresponds to replacing x by an r-dimensional subspace of $h^\perp_{\mathbf{Q}}$ (see [BLMM, Section 2.6]; the terminology there is *special cycles*).

2.3. **Lattice polarized hyperkähler varieties.** Fix a primitive embedding $\Sigma \hookrightarrow \Lambda$ of a lattice of signature $(1, \mathrm{rank}(\Sigma) - 1)$ and a primitive class $h \in \Sigma$ such that $h^2 > 0$. Following Dolgachev and Camere, one may consider (Σ, h)-polarized hyperkähler varieties (briefly, one considers polarized hyperkähler varieties (X, H) with an embedding $\Sigma \hookrightarrow \mathrm{Pic}(X)$ that maps h to H; see the discussion in [BL, Section 3.7]).

There is again a quasi-projective moduli space $\mathscr{F}_{\Lambda,\Sigma,h}$ for these, with a forgetful map $\mathscr{F}_{\Lambda,\Sigma,h} \to \mathscr{F}_{\Lambda,h}$. The period domain is $D_{\Sigma^\perp}$ and there is a period map $\mathscr{F}_{\Lambda,\Sigma,h} \to D_{\Sigma^\perp}/\Gamma_\Sigma$, where Γ_Σ is the stabilizer of Σ in the suitable monodromy group (which may depend on the component of $\mathscr{F}_{\Lambda,\Sigma,h}$). The Noether–Lefschetz r-cycles, as defined above, are the images of the maps $D_{\Sigma^\perp}/\Gamma_\Sigma \to D_{\Lambda,h}/\Gamma$ for all possible lattices Σ of rank $r+1$ as above.

One can also combine the two constructions and consider Noether–Lefschetz r-cycles in $D_{\Sigma^\perp}/\Gamma_\Sigma$ (geometrically, these correspond to (Σ, h)-polarized hyperkähler varieties with Picard group of rank at least $\mathrm{rank}(\Sigma) + r$).

The main theorem of [BLMM] mentioned above still applies and implies for example that Noether–Lefschetz divisors span the Picard group of $D_{\Sigma^\perp}/\Gamma_\Sigma$ (and of $\mathscr{F}_{\Lambda,\Sigma,h}$) when $s - \mathrm{rank}(\Sigma) \geq 2$ ([BL, Proposition 4.1.1]; Corollary 2.3).

2.4. **Cohomology classes of subvarieties of Shimura varieties.** One would like to associate cohomology classes for the special cycles constructed above in the Shimura variety Y_Γ (of orthogonal type $(2, s)$). There are two problems: Y_Γ is not smooth (this does not really matter, because it has finite quotient singularities, so I will not discuss that) and it is not compact. Fortunately, it has a compactification Y_Γ^{BB} with very small boundary (but also very singular).

There are various cohomology theories on Y_Γ (I reproduce here the discussion of [BLMM, Section 2.4]):

- the reduced L^2-cohomology $\bar{H}^\bullet_{L^2}(Y_\Gamma, \mathbf{C})$;
- the L^2-cohomology $H^\bullet_{L^2}(Y_\Gamma, \mathbf{C})$;
- the de Rham cohomology $H^\bullet_{\mathrm{dR}}(Y_\Gamma, \mathbf{C})$;
- the intersection cohomology $IH^\bullet(Y_\Gamma, \mathbf{C})$ on Y_Γ^{BB};

and maps

$$\alpha\colon \bar H^\bullet_{L^2}(Y_\Gamma,\mathbf{C})\to H^\bullet_{L^2}(Y_\Gamma,\mathbf{C})\to H^\bullet_{\mathrm{dR}}(Y_\Gamma,\mathbf{C})\quad\text{and}\quad \beta\colon H^\bullet_{L^2}(Y_\Gamma,\mathbf{C})\to IH^\bullet(Y_\Gamma,\mathbf{C}).$$

For the next theorem, I refer to [BLMM, Example 3.4].

Theorem 2.1. *In degrees $< s-1$, the maps α and β are isomorphisms and all these groups carry natural (isomorphic) pure Hodge structures.*

Furthermore, Kudla and Millson construct, for special r-cycles, cohomology classes that belong to $\alpha(\bar H^{2r}_{L^2}(Y_\Gamma,\mathbf{C}))\subset H^{2r}_{\mathrm{dR}}(Y_\Gamma,\mathbf{C})$. The main theorem of [BLMM] is then the following ([BLMM, Theorem 3.6]).

Theorem 2.2. *For $r<(s+1)/3$, the subspace $H^{r,r}(Y_\Gamma)\subset H^{2r}(Y_\Gamma,\mathbf{C})$ is defined over $\mathbf{Q}$ and spanned by classes of special r-cycles.*

For $r<s/4$, classes of special r-cycles span the whole of $H^{2r}(Y_\Gamma,\mathbf{Q})$.

Corollary 2.3. *When $s-\mathrm{rank}(\Sigma)\geq 2$, the Picard groups of $D_{\Sigma^\perp}/\Gamma_\Sigma$ and of $\mathscr{F}_{\Lambda,\Sigma,h}$ are spanned by classes of Noether–Lefschetz divisors.*

2.5. **Noether–Lefschetz divisors revisited.** I will explain here various labellings used for Noether–Lefschetz divisors, in particular in [MP]. The set up is still the same: Λ is a lattice with signature $(2,s)$ with a primitive class h such that $l:=h^2>0$, we define the period domain $D_{\Lambda,h}$ as in (1), and we work in the quasi-projective quotient $Y_\Gamma:=D_{\Lambda,h}/\Gamma$, where $\Gamma\subset \mathrm{O}(\Lambda,h)$ is some arithmetic subgroup.[4]

2.5.1. *Noether–Lefschetz divisors of the first type.* We consider the rank-2 lattice

$$\mathbb{L}_{m,d}:=\begin{pmatrix} l & d\\ d & 2m-2\end{pmatrix}$$

with the vector $v:=\begin{pmatrix}1\\0\end{pmatrix}$. We set

$$\Delta(m,d):=-\det(\mathbb{L}_{m,d})=d^2-2lm+2l,$$
$$\delta(m,d):=d\in(\mathbf{Z}/l\mathbf{Z})/\pm 1.$$

The pair $(\Delta(m,d),\delta(m,d))$ characterizes the isometry class of the pair $(\mathbb{L}_{m,d},v)$ so we will also write $\mathbb{L}_{\Delta,\delta}$ instead of $\mathbb{L}_{m,d}$. We consider the closure $P_{\Delta,\delta}$ in Y_Γ of the image of

$$\{[\omega]\in D_{\Lambda,h}\mid (\omega^\perp\cap\Lambda,h)\simeq(\mathbb{L}_{\Delta,\delta},v)\}$$

in Y_Γ (geometrically, points of this image correspond to polarized hyperkähler varieties (X,H) such that $(\mathrm{Pic}(X),H)\simeq(\mathbb{L}_{\Delta,\delta},v)$). This *Noether-Lefschetz divisor of the first type* is empty unless $\Delta>0$.

[4]Actually, Maulik and Pandaripandhe work on the moduli space of quasi-polarized K3 surfaces, where Λ is the unimodular K3 lattice and $\Gamma=\mathrm{O}(\Lambda,h)=\widetilde{O}(h^\perp)$ (see [MP, Section 4.3]). I am not sure in which generality this works.

2.5.2. *Noether–Lefschetz divisors of the second type.* Maulik and Pandaripandhe also define another type of Noether-Lefschetz divisors with better formal properties as follows: whenever $\Delta(m,d)>0$, they define $D_{m,d}$ as the image in Y_Γ of

$$\begin{aligned}&\{[\omega]\in D_{\Lambda,h}\mid \exists\beta\in(\omega^\perp\cap\Lambda,h)\quad \beta\cdot h=d,\ \beta^2=2m-2\}\\ =\ &\{[\omega]\in D_{\Lambda,h}\mid (\omega^\perp\cap\Lambda,h)\supset(\mathbb{L}_{m,d},v)\}.\end{aligned}$$

More precisely, they assign multiplicities as follows:

$$D_{m,d}:=\sum_{\Delta,\delta}\mu_{\Delta,\delta}P_{\Delta,\delta},$$

where

$$\mu_{\Delta,\delta}:=\operatorname{Card}\{\beta\in\mathbb{L}_{\Delta,\delta}\mid \beta\cdot h=d,\ \beta^2=2m-2\}\in\{0,1,2\}.$$

Note that $\mu_{\Delta,\delta}$ is in $\{0,1,2\}$ and that it is 0 unless $0<\Delta\mid\Delta(m,d)$ (because $\mathbb{L}_{\Delta,\delta}\supset\mathbb{L}_{m,d}$) hence the sum is finite.

2.5.3. *Heegner divisors.* Maulik and Pandaripandhe also use a third notation for Noether–Lefschetz divisors, which are then called Heegner divisors, in [MP, Section 4.3] (see also [BLMM, Section 2.6]). Take $n\in\mathbf{Q}^{>0}$ and $\gamma\in D(h^\perp)$ and consider the image $y_{n,\gamma}$ in Y_Γ of

$$\sum_{x^2=2n,\ x\equiv\gamma\in D(h^\perp)} D_x$$

(see (5) for the notation D_x). These are divisors in Y_Γ and $y_{n,\gamma}=y_{n,-\gamma}$; the divisor $y_{n,\gamma}$ has multiplicity 2 everywhere whenever $2\gamma\equiv 0$ in $D(h^\perp)$. Moreover, they set $y_{0,\gamma}=0$ except when $\gamma=0$, where it is the Hodge bundle.

In the K3 case, these Heegner divisors are exactly the Noether–Lefschetz divisors of the second type (this is [MP, Lemma 3]).

Lemma 2.4. *When Λ is the K3 lattice, one has $D(h^\perp)\simeq\mathbf{Z}/l\mathbf{Z}$ and $D_{m,d}=y_{n,\gamma}$, where*

$$n=-\frac{\Delta(m,d)}{2l}\quad\text{and}\quad \gamma\equiv d\in D(h^\perp)\simeq\mathbf{Z}/l\mathbf{Z}.$$

The proof is elementary.

2.5.4. *Hassett's Heegner divisors.* Finally, there is also Hassett's notation. He works on the moduli space of polarized hyperkähler varieties of $K3^{[m]}$-type and considers those whose Picard group contains a primitive, signature-$(1,1)$ lattice K containing the polarization. These form algebraic divisors in the moduli space and the Picard group of the hyperkähler variety corresponding to a very general point of this divisor is K. So these seem to be Noether–Lefschetz divisors of the first type. Hassett labels them by the even positive integer $d:=-\operatorname{disc}(K^\perp)$ (the idea is that $K^\perp$ can be isomorphic to the primitive lattice of a polarized K3 surface of degree d, which is then said to be associated with the hyperkähler variety).

Appendix A. Groups of lattice isometries

Let Λ be a lattice. The *discriminant group* $D(\Lambda)$ is the finite abelian group $\Lambda^\vee/\Lambda$, where $\Lambda\subset\Lambda^\vee\subset\Lambda_{\mathbf{Q}}$. There is a canonical morphism of groups $O(\Lambda)\to\operatorname{Aut}(D(\Lambda))$. When Λ is even, the quadratic form of Λ induces a $\mathbf{Q}/2\mathbf{Z}$-valued quadratic form on $D(\Lambda)$ and this map induces a group morphism

$$\pi\colon O(\Lambda)\longrightarrow \mathrm{O}(D(\Lambda))$$

which is surjective when the rank of Λ is at least the minimal number of generators of the group $D(\Lambda)$, plus 3 ([N, Remark 1.14.3]).

Let (r,s) be the signature of Λ and assume that both r and s are positive. There are surjective morphisms

$$O(r,s) \xrightarrow{\det} \{\pm 1\} \quad , \quad O(r,s) \xrightarrow{\mathrm{spn}} \{\pm 1\},$$

where spn is the *spinor norm*. The kernel of the first morphism is $\mathrm{SO}(r,s)$ and the kernel of the second morphism is denoted by $\mathrm{O}^+(r,s)$. The group $\mathrm{SO}^+(r,s) := \mathrm{SO}(r,s)\cap \mathrm{O}^+(r,s)$ is connected. One has $\det(-\mathrm{Id}) = (-1)^{r+s}$ and $\mathrm{spn}(-\mathrm{Id}) = (-1)^s$.

We define $\mathrm{SO}(\Lambda)$ and $\mathrm{O}^+(\Lambda)$ in an obvious way and we set

$$\widetilde{\mathrm{O}}(\Lambda) := \mathrm{Ker}(\pi) \quad , \quad \widehat{\mathrm{O}}(\Lambda) := \pi^{-1}(\pm\mathrm{Id}). \tag{6}$$

We define similarly $\widetilde{\mathrm{SO}}(\Lambda)$, $\widehat{\mathrm{SO}}(\Lambda)$, $\widetilde{\mathrm{O}}^+(\Lambda)$, $\widehat{\mathrm{O}}^+(\Lambda)$, $\widetilde{\mathrm{SO}}^+(\Lambda)$, $\widehat{\mathrm{SO}}^+(\Lambda)$.

When $r=2$ and Λ is an even lattice which splits as the orthogonal direct sum $U^{\oplus 2}\oplus M$, the isometry that exchanges the two copies of the hyperbolic plane U and is the identity on M is in $\widetilde{\mathrm{SO}}(\Lambda)$ but not in $\widetilde{\mathrm{SO}}^+(\Lambda)$. Therefore, the groups $\widetilde{\mathrm{O}}^+(\Lambda)$, $\widehat{\mathrm{O}}^+(\Lambda)$, $\widetilde{\mathrm{SO}}^+(\Lambda)$, $\widehat{\mathrm{SO}}^+(\Lambda)$ of isometries of spinor norm 1 all have index 2 in the groups $\widetilde{\mathrm{O}}(\Lambda)$, $\widehat{\mathrm{O}}(\Lambda)$, $\widetilde{\mathrm{SO}}(\Lambda)$, $\widehat{\mathrm{SO}}(\Lambda)$.

Appendix B. Known examples of hyperkähler varieties

We describe the known deformation types of hyperkähler varieties, their (even) Beauville–Bogomolov–Fujiki lattice Λ, discriminant groups, and monodromy groups. The notation for the various groups of isometries was defined in (6) and $\rho(n)$ is the number of prime factors of the positive integer n (so that $\rho(1)=0$). A useful reference is [R].

B.1. **K3 surfaces.** One has ([H, Section 6.3])

$$\begin{aligned}\Lambda &= E_8(-1)^{\oplus 2}\oplus U^{\oplus 3}, \quad \text{signature } (3,19),\\ D(\Lambda) &= 0, \quad \mathrm{O}(D(L)) = \{1\},\\ \mathrm{Mon}(\Lambda) &= \mathrm{O}^+(\Lambda).\end{aligned}$$

B.2. **Hyperkähler varieties of $K3^{[m]}$-type, $m\geq 2$.** One has ([M1, Lemma 4.2], [M2, Lemma 9.2])

$$\begin{aligned}\Lambda &= E_8(-1)^{\oplus 2}\oplus U^{\oplus 3}\oplus(-(2m-2)), \quad \text{signature } (3,20),\\ D(\Lambda) &= \mathbf{Z}/(2m-2)\mathbf{Z}, \quad q(\bar{1}) = -\tfrac{1}{2m-2}, \quad \mathrm{O}(D(L)) = (\mathbf{Z}/2\mathbf{Z})^{\rho(m-1)},\\ \mathrm{Mon}(\Lambda) &= \widehat{\mathrm{O}}^+(\Lambda), \quad \text{of index } 2^{\max(0,\rho(m-1)-1)} \text{ in } \mathrm{O}^+(\Lambda).\end{aligned}$$

B.3. **Hyperkähler varieties of $\mathrm{Kum}^{[m]}$-type, $m\geq 2$.** One has ([Mo])

$$\begin{aligned}\Lambda &= U^{\oplus 3}\oplus(-(2m+2)), \quad \text{signature } (3,4),\\ D(\Lambda) &= \mathbf{Z}/(2m+2)\mathbf{Z}, \quad q(\bar{1}) = -\tfrac{1}{2m+2}, \quad \mathrm{O}(D(L)) = (\mathbf{Z}/2\mathbf{Z})^{\rho(m+1)},\\ \mathrm{Mon}(\Lambda) &\subset \widetilde{\mathrm{O}}^+(\Lambda), \quad \text{of index 2, hence of index } 2^{\rho(m+1)+1} \text{ in } \mathrm{O}^+(\Lambda).\end{aligned}$$

B.4. **Hyperkähler varieties of OG10-type.** One has ([O][5])

$$\Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} \oplus \left(\begin{smallmatrix} -6 & 3 \\ 3 & -2 \end{smallmatrix}\right), \quad \text{signature } (3,21),$$
$$D(\Lambda) = \mathbf{Z}/3\mathbf{Z}, \quad q(\bar{1}) = -\tfrac{2}{3}, \quad \mathrm{O}(D(L)) = \mathbf{Z}/2\mathbf{Z},$$
$$\mathrm{Mon}(\Lambda) = \mathrm{O}^+(\Lambda) = \widehat{\mathrm{O}}^+(\Lambda).$$

B.5. **Hyperkähler varieties of OG6-type.** One has ([MR])

$$\Lambda = U^{\oplus 3} \oplus (-2)^{\oplus 2}, \quad \text{signature } (3,5),$$
$$D(\Lambda) = (\mathbf{Z}/2\mathbf{Z})^2, \quad q = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \mathrm{O}(D(L)) = (\mathbf{Z}/2\mathbf{Z})^2,$$
$$\mathrm{Mon}(\Lambda) = \mathrm{O}^+(\Lambda) = \widehat{\mathrm{O}}^+(\Lambda).$$

As explained in [H, Corollary 6.3], when $\mathrm{Mon}(\Lambda) = \mathrm{O}^+(\Lambda)$, the birational Torelli theorem holds: hyperkähler varieties (of that deformation type) are bimeromorphically isomorphic if and only if their integral Hodge structures are isomorphic.

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[5]The statement made in [Mo] about the monodromy group is wrong.